January 14 Math 2306 sec. 54 Spring 2019

Section 2: Initial Value Problems

An initial value problem consists of an ODE with a certain type of additional conditions.

Solve the equation ¹

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}) \tag{1}$$

subject to the initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, y^{(n-1)}(x_0) = y_{n-1}.$$
 (2)

The problem (1)–(2) is called an *initial value problem* (IVP).

January 11, 2019 1 / 23

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¹on some interval *I* containing x_0 .

Example

Part 1: Verify that $x = c_1 \cos(2t) + c_2 \sin(2t)$ is a 2-parameter family of solutions of the ODE

$$x''+4x=0.$$

$$X = C_1 C_0 r (2t) + C_2 Sin (2t)$$

$$X' = -2C_1 Sin (2t) + 2C_2 C_0 s (2t)$$

$$X'' = -4C_1 C_0 r (2t) - 4C_2 Sin (2t)$$

Substitute

 $x''+4x = -4c_1 \cos(2t) - 4c_2 \sin(2t) + 4(c_1 \cos(2t) + c_2 \sin(2t)) =$

- =-4c, G1(2+)-4cz sin(2+)+4c, Gr(2+)+4cz sin(2+)
 - = $C_{ss}(2t) \left(-4c_1 + 4c_1\right) + Sin(2t) \left(-4c_2 + 4c_2\right)$
 - = 0 + 0 = 0
- So X= C, Cor(2t) + C2 Sin(2t) is a 2-parameter family of solutions.

Example

Part 2: Find a solution of the IVP

$$x'' + 4x = 0, \quad x\left(\frac{\pi}{2}\right) = -1, \quad x'\left(\frac{\pi}{2}\right) = 4$$

$$\text{lie know that } x = c_1 \operatorname{Cor}(2t) + c_2 \operatorname{Sin}(2t), \quad \text{Impose}$$

$$\text{the initial conditions}, \quad x' = -2c_1 \operatorname{Sin}(2t) + 2c_2 \operatorname{Cos}(2t)$$

$$x\left(\frac{\pi}{2}\right) = c_1 \operatorname{Cos}(2 \cdot \frac{\pi}{2}) + c_2 \operatorname{Sin}(2 \cdot \frac{\pi}{2}) = -1$$

$$c_1(-1) + c_2(0) = -1 \implies c_1 = 1$$

January 11, 2019 4 / 23

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$$X'(=) = -2C_{1}S_{1}n(2\cdot=) + 2C_{2}G_{2}(2\cdot=) = 4$$

-2C_{1}(0) + 2C_{2}(-1) = 4 = C_{2} = -2

January 11, 2019 5 / 23

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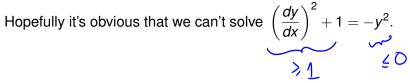
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Existence and Uniqueness

Two important questions we can always pose (and sometimes answer) are

(1) Does an IVP have a solution? (existence) and

(2) If it does, is there just one? (uniqueness)



January 11, 2019

6/23

Uniqueness

Consider the IVP

$$\frac{dy}{dx} = x\sqrt{y}$$
 $y(0) = 0$

Verify that $y = \frac{x^4}{16}$ is a solution of the IVP. And find a second solution of the IVP by clever guessing.

Solving an IVP means satisfying a differential equation AND satisfying an initial condition.

Show that $y = \frac{x^4}{16}$ satisfies the initial condition: when x=0, y=0 $y(0) = \frac{0^4}{16} = \frac{0}{16} = 0$ y due satisfy the I.C.

January 11, 2019 7 / 23

$$\frac{dy}{dx} = x\sqrt{y} \quad y(0) = 0$$

Show that $y = \frac{x^4}{16}$ solves the differential equation: $y = \frac{x^{4}}{l_{b}} \Rightarrow \frac{dy}{l_{c}} = \frac{4x^{3}}{l_{b}} = \frac{x^{3}}{4} \text{ and } \sqrt{y} = \sqrt{\frac{x^{4}}{l_{b}}} = \frac{1x^{2}}{4} = \frac{x^{2}}{4}$ $\frac{dy}{dx} = x\sqrt{y}$ $\frac{x^{3}}{y} = x\left(\frac{x^{2}}{y}\right) \Rightarrow \frac{x^{3}}{y} = \frac{x^{3}}{y} + r^{2}x$ y= X4 is a solution to the IVP

$$\frac{dy}{dx} = x\sqrt{y} \quad y(0) = 0$$

Find another solution to the IVP. (Hint: Think about really simple functions like a constant function.)

$$y(x) = 0$$
 is a solution.
Note $y = 0 \Rightarrow \frac{dy}{dx} = 0$ and $x\sqrt{y} = x\sqrt{0} = 0$

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Section 3: Separation of Variables

The simplest type of equation we could encounter would be of the form

$$\frac{dy}{dx} = g(x).$$

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For example, solve the ODE

January 11, 2019 10 / 23

Separable Equations

Definition: The first order equation y' = f(x, y) is said to be **separable** if the right side has the form

$$f(x,y)=g(x)h(y).$$

That is, a separable equation is one that has the form

$$\frac{dy}{dx}=g(x)h(y).$$

January 11, 2019

11/23

Determine which (if any) of the following are separable.

(a)
$$\frac{dy}{dx} = x^3 y$$
 This is separable.
 $g(x) = x^3$ and $h(y) = y$

(b)
$$\frac{dy}{dx} = 2x + y$$
 This is not separable.
Note for example $2x + y = x\left(2 + \frac{y}{x}\right)$