January 14 Math 2306 sec 58 Spring 2016

Section 1: Concepts and Terminology

Solution of $F(x, y, y', \dots, y^{(n)}) = 0$ (*)

Definition: A function ϕ defined on an interval I[1] and possessing at least n continuous derivatives on I is a **solution** of (*) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

Definition: An **implicit solution** of (*) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (*) and this relation.

[1] The interval is called the *domain of the solution* or the *interval of definition*.

Examples:

Verify that the given function is an solution of the ODE on the indicated interval.

$$\phi(x) = 5\tan(5x), \quad I = \left(-\frac{\pi}{10}, \frac{\pi}{10}\right), \quad y' - 25 = y^2$$

Recall: $\tan\theta$ is cont. and differentiable if

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ If $-\frac{\pi}{2} < 5x < \frac{\pi}{2}$ then

$-\frac{\pi}{10} < x < \frac{\pi}{10}$

So ϕ is differentiable on $\left(-\frac{\pi}{10}, \frac{\pi}{10}\right)$.

Let $y = 5\tan(5x)$ then $y' = 5\sec^2(5x)\cdot 5 = 25\sec^2(5x)$

The ODE is $y' - 25 = y^2$

$$y' - 25 = 25\sec^2(5x) - 25 \stackrel{?}{=} y^2 = \left(5\tan(5x)\right)^2$$

$$25\left(\sec^2(5x) - 1\right) \stackrel{?}{=} 25\tan^2(5x)$$

Recall $\sec^2\theta = \tan^2\theta + 1 \Rightarrow \sec^2\theta - 1 = \tan^2\theta$

So

$$25\tan^2(5x) = 25\tan^2(5x)$$

This is an identity (i.e. true for all x in I)

Examples:

Verify that the relation defines and implicit solution of the differential equation.

$$y^2 - 2x^2y = 1, \quad 2xy\,dx + (x^2 - y)\,dy = 0$$

We can write the DE in normal form:

$$(x^2 - y)\,dy = -2xy\,dx \implies \frac{dy}{dx} = \frac{-2xy}{x^2 - y} \quad \text{for } x^2 - y \neq 0$$

We'll use implicit differentiation to find $\frac{dy}{dx}$ from

$$y^2 - 2x^2y = 1$$

$$2y\frac{dy}{dx} - 2\left(2xy + x^2\frac{dy}{dx}\right) = 0$$

$$y\frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} = 0$$

$$(y-x^2)\frac{dy}{dx} = 2xy$$

$$\frac{dy}{dx} = \frac{2xy}{y-x^2} = \frac{-2xy}{x^2-y}$$

which matches the given ODE.

Function vs Solution

The interval of defintion has to be an interval.

Consider $y' = -y^2$. Clearly $y = \frac{1}{x}$ solves the DE. The interval of defintion can be $(-\infty, 0)$, or $(0, \infty)$—or any interval that doesn't contain the origin. But it can't be $(-\infty, 0) \cup (0, \infty)$ because this isn't an interval!

Often, we'll take I to be the largest, or one of the largest, possible interval. It may depend on other information.

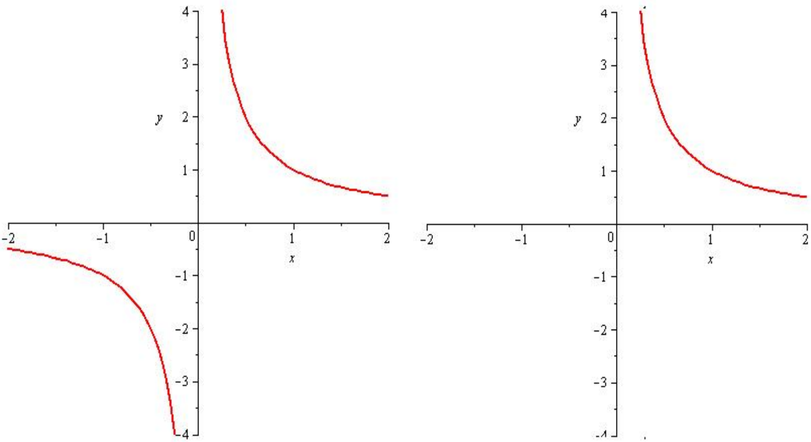


Figure: Left: Plot of $f(x) = \frac{1}{x}$ as a **function**. Right: Plot of $f(x) = \frac{1}{x}$ as a possible **solution** of an ODE.

Solutions with Parameters (unspecified constants)

Show that for any choice of constants c_1 and c_2, $y = c_1 x + \frac{c_2}{x}$ is a solution of the differential equation

$$x^2 y'' + xy' - y = 0$$

$$y = c_1 x + \frac{c_2}{x}$$

$$y' = c_1 + \frac{-c_2}{x^2}$$

$$y'' = \frac{2c_2}{x^3}$$

$$x^2 y'' + xy' - y =$$

$$x^2\left(\frac{2c_2}{x^3}\right) + x\left(c_1 - \frac{c_2}{x^2}\right) - \left(c_1 x + \frac{c_2}{x}\right) =$$

$$\cancel{\frac{2c_2}{x}} + \cancel{c_1} x - \cancel{\frac{c_2}{x}} - \cancel{c_1} x - \cancel{\frac{c_2}{x}}$$

$$= 0 \quad \text{as required.}$$

Some Terms

- A **parameter** is an unspecified constant such as c_1 and c_2 in the last example.
- A **family of solutions** is a collection of solution functions that only differ by a parameter.
- An ***n*-parameter family of solutions** is one containing n parameters (e.g. $c_1 x + \frac{c_2}{x}$ is a 2 parameter family).
- A **particular solution** is one with no arbitrary constants in it.
- The **trivial solution** is the simple constant function $y = 0$.
- An **integral curve** is the graph of one solution (perhaps from a family).

Section 2: Initial Value Problems IVP

An initial value problem consists of an ODE with additional conditions.

Solve the equation [2]

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}) \tag{1}$$

subject to the *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, y^{(n-1)}(x_0) = y_{n-1}. \tag{2}$$

The problem (1)–(2) is called an *initial value problem* (IVP).

y and it's derivatives are all given at the same x_0

[2] on some interval I containing x_0.

First order case:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

1st order ODE

one condition

Second order case:

$$\frac{d^2y}{dx^2} = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

2nd order ODE

2 conditions

Example

Given that $y = c_1 x + \frac{c_2}{x}$ is a 2-parameter family of solutions of $x^2y'' + xy' - y = 0$, solve the IVP

$$x^2y'' + xy' - y = 0, \quad y(1) = 1, \quad y'(1) = 3$$

Since all solutions look like $y = c_1 x + \frac{c_2}{x}$ we need to find c_1, c_2 that satisfy the Initial conditions.

Impose $y(1) = 1$ $\quad y(1) = c_1(1) + \frac{c_2}{1} = 1$

i.e. $c_1 + c_2 = 1$

Recall $y' = c_1 - \frac{c_2}{x^2}$

Impose $y'(1)=3$ $\quad y'(1) = c_1 - \frac{c_2}{1^2} = 3$

i.e. $c_1 - c_2 = 3$

We need $\left.\begin{array}{l} c_1 + c_2 = 1 \\ c_1 - c_2 = 3 \end{array}\right\} \xRightarrow{\text{add}}$ $2c_1 = 4 \Rightarrow c_1 = 2$

sub into eqn. 1

$2 + c_2 = 1 \Rightarrow c_2 = -1$

The solution to the IVP is $\boxed{y = 2x - \frac{1}{x}}$.

Note: this is a particular solution

Example

Part 1

Show that for any constant c the relation $x^2 + y^2 = c$ is an implicit solution of the ODE

$$\frac{dy}{dx} = -\frac{x}{y}$$

Use implicit differentiation: $x^2 + y^2 = c$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{-2x}{2y} \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

Example

Part 2

Use the preceding results to find an explicit solution of the IVP

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(0) = -2$$

From part 1, solutions satisfy the relation

$$x^2 + y^2 = C$$

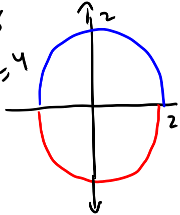
The condition $y(0) = -2$ tells us that the point $(0, -2)$ is on the graph of y.

So $\quad 0^2 + (-2)^2 = C \Rightarrow C = 4$

So our solution solves

$$x^2 + y^2 = 4$$

2 functions solve $x^2+y^2=4$



Let's solve for y: $\quad y^2 = 4 - x^2$

we have 2 possibilities $\quad y = \sqrt{4-x^2}$ or

$$y = -\sqrt{4-x^2}$$

Since $y(0) = -2$, the bottom one is correct

$$y = -\sqrt{4-x^2}.$$

This is the bottom half circle

Graphical Interpretation

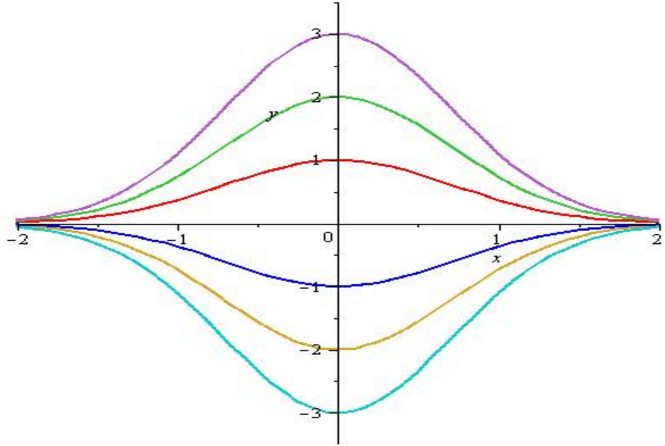


Figure: Each curve solves $y' + 2xy = 0$, $y(0) = y_0$. Each colored curve corresponds to a different value of y_0

Example

$x = c_1\cos(2t) + c_2\sin(2t)$ is a 2-parameter family of solutions of the ODE $x'' + 4x = 0$. Find a solution of the IVP

$$x'' + 4x = 0, \quad x\left(\frac{\pi}{2}\right) = -1, \quad x'\left(\frac{\pi}{2}\right) = 4$$

The solution has the form $x = c_1 \cos 2t + c_2 \sin 2t$

$$x' = -2c_1 \sin 2t + 2c_2 \cos 2t$$

Impose $x\left(\frac{\pi}{2}\right) = -1$ $\quad x\left(\frac{\pi}{2}\right) = c_1 \cos\left(2\cdot\frac{\pi}{2}\right) + c_2 \sin\left(2\cdot\frac{\pi}{2}\right) = -1$

$$-c_1 + c_2\cdot 0 = -1 \Rightarrow c_1 = 1$$

Impose $x'\left(\frac{\pi}{2}\right) = 4$

$$x'\left(\frac{\pi}{2}\right) = -2\sin\left(2\cdot\frac{\pi}{2}\right) + 2C_2\cos\left(2\cdot\frac{\pi}{2}\right) = 4$$

$$0 + 2C_2\cdot(-1) = 4 \Rightarrow C_2 = -2$$

The solution to the IVP is

$$x = \cos 2t - 2\sin 2t\,.$$

Existence and Uniqueness

Two important questions we can always pose (and sometimes answer) are

(1) Does an IVP have a solution? (existence) and

(2) If it does, is there just one? (uniqueness)

Hopefully it's obvious that we can't solve $\left(\frac{dy}{dx}\right)^2 + 1 = -y^2.$

Uniqueness

Consider the IVP

$$\frac{dy}{dx} = x\sqrt{y} \quad y(0) = 0$$

Verify that $y = \frac{x^4}{16}$ is a solution of the IVP. And find a second solution of the IVP by clever guessing.

Let's verify that $y = \frac{x^4}{16}$ solves the IVP.

- It satisfies $y(0) = 0$? $y(0) = \frac{0^4}{16} = 0$ yes it does.
- It satisfies the ODE?

$$y = \frac{x^4}{16} \Rightarrow y' = \frac{4x^3}{16} = \frac{x^3}{4}$$

Note $x\sqrt{y} = x\sqrt{\frac{x^4}{16}} = x\,\frac{x^2}{4} = \frac{x^3}{4}$

so $\frac{dy}{dx} = \frac{x^3}{4} = x\sqrt{y}$

So $y = \frac{x^4}{16}$ solves the IVP.

2^{nd} soln: $\frac{dy}{dx} = x\sqrt{y}\,, \quad y(0)=0$

Is there a constant soln? Does $y=0$ solve it?

If $y=0$, then $y(0)=0$. And $\frac{dy}{dx} = 0 = x\sqrt{0}$

A second solution is the constant $y=0$.