January 14 Math 2306 sec 59 Spring 2016

Section 1: Concepts and Terminology

Solution of $F(x, y, y', \ldots, y^{(n)}) = 0$ (*)

Definition: A function ϕ defined on an interval I[1] and possessing at least n continuous derivatives on I is a **solution** of (*) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

Definition: An **implicit solution** of (*) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (*) and this relation.

[1] The interval is called the *domain of the solution* or the *interval of definition*.

Examples:

Verify that the given function is an solution of the ODE on the indicated interval.

$$\phi(t) = 3e^{2t}, \quad I = (-\infty, \infty)\,, \quad \frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$$

Note that $\phi(t) = 3e^{2t}$ has at least 2 derivatives on $(-\infty, \infty)$.

Set $y = \phi(t)$ i.e. $y = 3e^{2t}$

Note $y' = 6e^{2t}$ and $y'' = 12e^{2t}$

Substitute $\quad y'' - y' - 2y =$

$$12e^{2t} - 6e^{2t} - 2(3e^{2t}) =$$

$$12e^{2t} - 6e^{2t} - 6e^{2t} = 0$$

Hence $\quad \phi(t) = 3e^{2t}$ is a solution.

Examples:

$$\phi(x) = 5\tan(5x), \quad I = \left(-\frac{\pi}{10}, \frac{\pi}{10}\right), \quad y' - 25 = y^2$$

$\tan\theta$ is continuous and differentiable if

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ Note: if $\frac{-\pi}{10} < x < \frac{\pi}{10}$

then $5 \cdot \frac{-\pi}{10} < 5x < 5 \cdot \frac{\pi}{10}$

i.e. $\frac{-\pi}{2} < 5x < \frac{\pi}{2}$

So ϕ is at least one time differentiable on I.

Set $y = 5\tan(5x) \implies y' = 5\sec^2(5x)\cdot 5 = 25\sec^2(5x)$

Substitute

$$y' - 25 = 25\sec^2(5x) - 25 \stackrel{?}{=} y^2 = (5\tan(5x))^2$$

$$25(\sec^2(5x) - 1) \stackrel{?}{=} 25\tan^2(5x)$$

Recall that $\tan^2\theta + 1 = \sec^2\theta$

$\implies \tan^2\theta = \sec^2\theta - 1$

Thus $y' - 25 = 25\tan^2(5x) = y^2$ as required

Examples:

Verify that the relation defines and implicit solution of the differential equation.

$$y^2 - 2x^2y = 1, \quad 2xy\,dx + (x^2 - y)\,dy = 0$$

We'll put the ODE in normal form:

$$(x^2-y)\,dy = -2xy\,dx \implies \frac{dy}{dx} = \frac{-2xy}{x^2-y} \quad \text{for } x^2-y \neq 0$$

Let's find $\frac{dy}{dx}$ from $y^2 - 2x^2y = 1$ by implicit differentiation.

$$2y\frac{dy}{dx} - 2\left(2xy + x^2\frac{dy}{dx}\right) = 0$$

Isolate $\frac{dy}{dx}$:

$$y\frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} = 0$$

$$(y - x^2)\frac{dy}{dx} = 2xy$$

$$\frac{dy}{dx} = \frac{2xy}{y - x^2}$$ for $y - x^2 \neq 0$

$$\frac{dy}{dx} = \frac{-2xy}{x^2 - y}$$ which matches the ODE.

Function vs Solution

The interval of defintion has to be an interval.

Consider $y' = -y^2$. Clearly $y = \frac{1}{x}$ solves the DE. The interval of defintion can be $(-\infty, 0)$, or $(0, \infty)$—or any interval that doesn't contain the origin. But it can't be $(-\infty, 0) \cup (0, \infty)$ because this isn't an interval!

Often, we'll take I to be the largest, or one of the largest, possible interval. It may depend on other information.

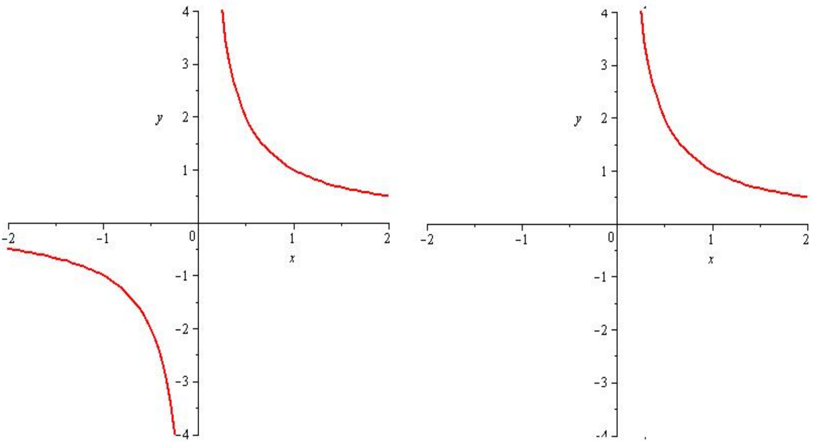


Figure: Left: Plot of $f(x) = \frac{1}{x}$ as a **function**. Right: Plot of $f(x) = \frac{1}{x}$ as a possible **solution** of an ODE.

Solutions with Parameters (unspecified constants)

Show that for any choice of constants c_1 and c_2, $y = c_1 x + \frac{c_2}{x}$ is a solution of the differential equation

$$x^2 y'' + xy' - y = 0$$

Sub. into the ODE

$$y = c_1 x + \frac{c_2}{x}$$

$$y' = c_1 - \frac{c_2}{x^2}$$

$$y'' = \frac{2c_2}{x^3}$$

$$x^2 y'' + x y' - y =$$

$$x^2\left(\frac{2c_2}{x^3}\right) + x\left(c_1 - \frac{c_2}{x^2}\right) - \left(c_1 x + \frac{c_2}{x}\right) =$$

$$\cancel{\frac{2c_2}{x}} + \cancel{c_1 x} - \cancel{\frac{c_2}{x}} - \cancel{c_1 x} - \cancel{\frac{c_2}{x}} =$$

$$0 = 0$$

Some Terms

- A **parameter** is an unspecified constant such as c_1 and c_2 in the last example.
- A **family of solutions** is a collection of solution functions that only differ by a parameter.
- An ***n*-parameter family of solutions** is one containing n parameters (e.g. $c_1 x + \frac{c_2}{x}$ is a 2 parameter family).
- A **particular solution** is one with no arbitrary constants in it.
- The **trivial solution** is the simple constant function $y = 0$.
- An **integral curve** is the graph of one solution (perhaps from a family).

Section 2: Initial Value Problems IVP

An initial value problem consists of an ODE with additional conditions.

Solve the equation [2]

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}) \tag{1}$$

subject to the *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, y^{(n-1)}(x_0) = y_{n-1}. \tag{2}$$

The problem (1)–(2) is called an *initial value problem* (IVP).

Note that $y, y', \dots, y^{(n-1)}$ are all given at the same $x = x_0$.

[2] on some interval I containing x_0.

First order case:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

1st order ODE

one condition

Second order case:

$$\frac{d^2y}{dx^2} = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

2nd order ODE

2 conditions @ the same input x

Example

Given that $y = c_1 x + \frac{c_2}{x}$ is a 2-parameter family of solutions of $x^2 y'' + xy' - y = 0$, solve the IVP

$$x^2 y'' + xy' - y = 0, \quad y(1) = 1, \quad y'(1) = 3$$

The solution has the form $y = c_1 x + \frac{c_2}{x}$

recall $y' = c_1 - \frac{c_2}{x^2}$

Impose the condition $y(1) = 1$

$$y(1) = c_1 \cdot 1 + \frac{c_2}{1} = 1 \Rightarrow c_1 + c_2 = 1$$

Impose the condition $y'(1) = 3$

$$y'(1) = c_1 - \frac{c_2}{1^2} = 3 \implies c_1 - c_2 = 3$$

We require $c_1 + c_2 = 1$

$c_1 - c_2 = 3$ $\Big\} \Rightarrow$ add

$2c_1 = 4 \quad c_1 = 2$

Sub into eqn. 1

$2 + c_2 = 1 \implies c_2 = -1$

$$\boxed{\text{The solution to the IVP is } \quad y = 2x - \frac{1}{x}.}$$

This is an example of a particular solution.

Example

Part 1

Show that for any constant c the relation $x^2 + y^2 = c$ is an implicit solution of the ODE

$$\frac{dy}{dx} = -\frac{x}{y}$$

Find $\frac{dy}{dx}$ using implicit diff: $x^2+y^2=c$

$$2x + 2y\frac{dy}{dx} = 0 \implies 2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y} \implies \frac{dy}{dx} = \frac{-x}{y}$$

So the relation defines an implicit solution.

Example

Part 2

Use the preceding results to find an explicit solution of the IVP

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(0) = -2$$

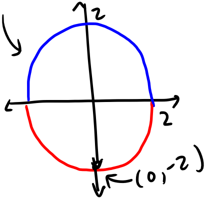
From part 1, if y solves the ODE then

$$x^2 + y^2 = C$$

$y(0) = -2$ implies that $0^2 + (-2)^2 = C$

$$4 = C$$

Our solution must satisfy $x^2+y^2=4$



Find an explicit solution:

$$y^2 = 4-x^2$$

So either $y=\sqrt{4-x^2}$ or $y=-\sqrt{4-x^2}$

Since $y(0)=-2$ our solution to the IVP is
$$\boxed{y=-\sqrt{4-x^2}\,.}$$

Graphical Interpretation

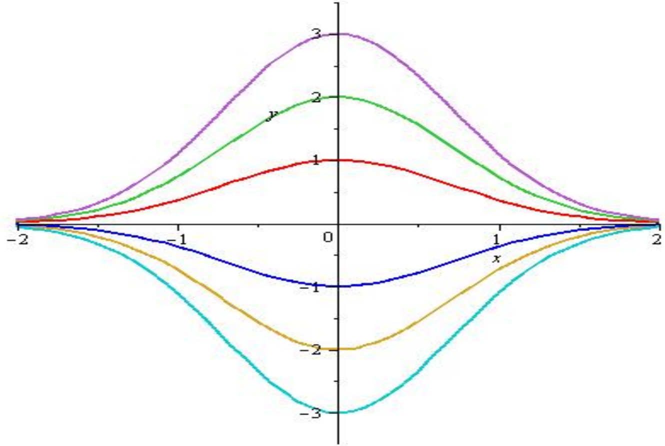


Figure: Each curve solves $y' + 2xy = 0$, $y(0) = y_0$. Each colored curve corresponds to a different value of y_0

Example

$x = c_1 \cos(2t) + c_2 \sin(2t)$ is a 2-parameter family of solutions of the ODE $x'' + 4x = 0$. Find a solution of the IVP

$$x'' + 4x = 0, \quad x\left(\frac{\pi}{2}\right) = -1, \quad x'\left(\frac{\pi}{2}\right) = 4$$

All solutions look like $x = c_1 \cos 2t + c_2 \sin 2t$

so $x' = -2c_1 \sin 2t + 2c_2 \cos 2t$

Impose $x\left(\frac{\pi}{2}\right) = -1$

$$x\left(\frac{\pi}{2}\right) = c_1 \cos\left(2 \cdot \frac{\pi}{2}\right) + c_2 \sin\left(2 \cdot \frac{\pi}{2}\right) = -1$$

$$-c_1 + 0 = -1 \quad \Rightarrow \quad c_1 = 1$$

Impose $x'\left(\frac{\pi}{2}\right) = 4$

$$x'\left(\frac{\pi}{2}\right) = -2\cdot 1 \sin\left(2\cdot\frac{\pi}{2}\right) + 2c_2\cos\left(2\cdot\frac{\pi}{2}\right) = 4$$

$$-2c_2 = 4 \quad \Rightarrow \quad c_2 = -2$$

The solution to the IVP is

$$x = \cos 2t - 2\sin 2t\,.$$