Jan. 15 Math 2254H sec 015H Spring 2015

Section 6.6: Inverse Trigonometric Functions

Recall that the inverse sine function is defined by

$$\sin^{-1}(x) = y \quad \text{if and only if} \quad x = \sin y \quad \text{and} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

$\sin^{-1}(x) = \arcsin(x)$

$$\sin^{-1}(\sin x) = x \quad \text{for} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$\sin\left(\sin^{-1} x\right) = x \quad \text{for} \quad -1 \le x \le 1$$

$\sin^{-1}(x) = y$ if and only if $x = \sin y$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

Evaluate if possible

(a) $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

Since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

and $\frac{-\pi}{2} \leq \frac{\pi}{6} \leq \frac{\pi}{2}$

(b) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{-\pi}{3}$

as $\sin\left(\frac{-\pi}{3}\right) = \frac{-\sqrt{3}}{2}$

and $\frac{-\pi}{2} \leq \frac{-\pi}{3} \leq \frac{\pi}{2}$

$\sin^{-1}(x) = y$ if and only if $x = \sin y$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

Evaluate if possible

(c) $\sin^{-1}(2)$ is undefined since $\sin\theta \leq 1$

for all θ

(d) $\sin^{-1}\left(\sin\frac{4\pi}{3}\right) = \sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \frac{-\pi}{3}$

Arccosine

Similarly

$$\cos^{-1}(x) = y \quad \text{if and only if} \quad x = \cos y \quad \text{and} \quad 0 \le y \le \pi$$

$$\cos^{-1}(\cos x) = x \quad \text{for} \quad 0 \le x \le \pi$$
$$\cos\left(\cos^{-1} x\right) = x \quad \text{for} \quad -1 \le x \le 1$$

$\cos^{-1}(x) = y$ if and only if $x = \cos y$ and $0 \le y \le \pi$

Evaluate if possible

(a) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

(b) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$

Since $\cos\left(\frac{3\pi}{4}\right) = \frac{-1}{\sqrt{2}}$

and

$0 \le \frac{3\pi}{4} \le \pi$

$\cos^{-1}(x) = y$ if and only if $x = \cos y$ and $0 \leq y \leq \pi$

Evaluate if possible

(c) $\cos^{-1}(-4)$ Doesn't exist

(d) $\cos^{-1}\left(\cos\frac{5\pi}{4}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}$

Arctangent

$$\tan^{-1}(x) = y \quad \text{if and only if} \quad x = \tan y \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\tan^{-1}(\tan x) = x \quad \text{for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\tan\left(\tan^{-1} x\right) = x \quad \text{for} \quad -\infty < x < \infty$$

$\tan^{-1}(x) = y$ if and only if $x = \tan y$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Evaluate if possible

(a) $\tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$

(b) $\tan^{-1}(-1) = -\frac{\pi}{4}$

$\tan^{-1}(x) = y$ if and only if $x = \tan y$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Evaluate if possible

(c) $\tan^{-1}\left(\tan\frac{7\pi}{4}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$

$\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = -\frac{\pi}{4}$

as well, so the result is not just due to $\frac{7\pi}{4}$ and $-\frac{\pi}{4}$ being coterminal

$\tan^{-1}(x) = y$ if and only if $x = \tan y$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Evaluate if possible

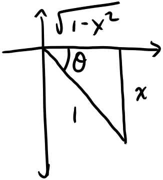
(d) $\tan\left(\sin^{-1} x\right)$

$= \dfrac{x}{\sqrt{1-x^2}}$

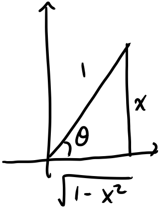
Let $\theta = \sin^{-1} x$

so $\sin\theta = x$

and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$



or



In both cases

$\tan\theta = \dfrac{x}{\sqrt{1-x^2}}$

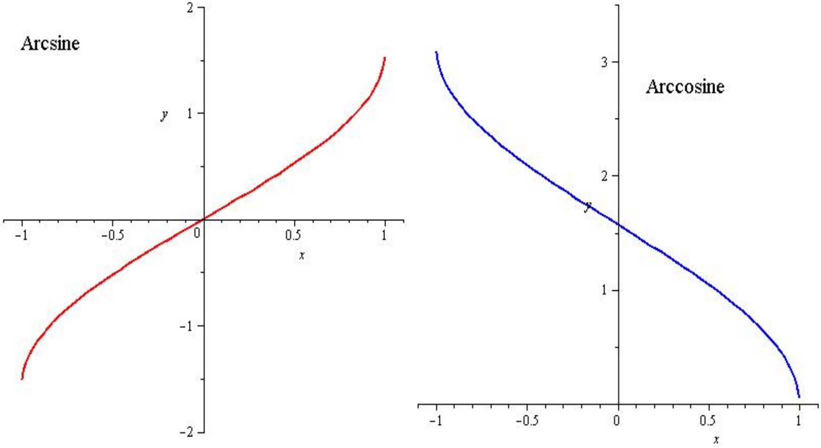


Figure: Plots of the inverse sine and cosine functions.

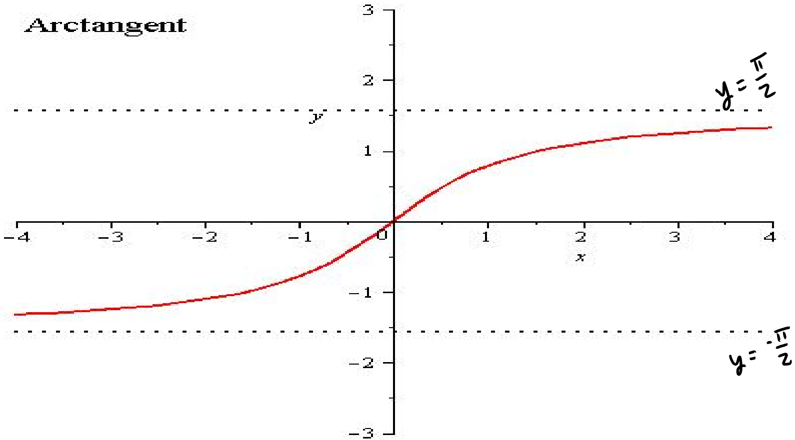


Figure: Plot of the arctangent function.

Important Arctangent limits

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}, \quad \text{and} \quad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

Example: Compute the limit

$$\lim_{t\to\infty} \tan^{-1}(e^t) = \frac{\pi}{2}$$

Since $\lim_{t\to\infty} e^t = \infty$

Other Inverse Trig Functions (defined ala Stewart)[1]

$$y = \csc^{-1} x \iff x = \csc y \text{ and } y \in (0, \pi/2] \cup (\pi, 3\pi/2], \ |x| \geq 1$$

$$y = \sec^{-1} x \iff x = \sec y \text{ and } y \in [0, \pi/2) \cup [\pi, 3\pi/2), \ |x| \geq 1$$

$$y = \cot^{-1} x \iff x = \cot y \text{ and } 0 < y < \pi, \ -\infty < x < \infty$$

[1] Unfortunately, different authors may define the range of the inverse secant and cosecant differently. We'll just have to accept this fact. The symbol $\in$ means *"is an element of."*

Differentiation

Use the relationship $\sin(\sin^{-1} x) = x$ to derive the differentiation rule

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}.$$

$$\sin(\sin^{-1} x) = x$$

$$\frac{d}{dx}\left[\sin(\sin^{-1} x)\right] = \frac{d}{dx}[x]$$

$$\cos(\sin^{-1} x)\,\frac{d}{dx}\sin^{-1} x = 1$$

$$\Longrightarrow$$

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\cos(\sin^{-1}x)}$$

* From the diagram on slide 10

$$\cos\left(\sin^{-1}x\right) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

So
$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \quad \text{for} \quad |x|<1$$

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}, \qquad \frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}, \qquad \frac{d}{dx}\csc^{-1}x = -\frac{1}{x\sqrt{x^2-1}}$$

Range Definitions & Derivative Rules

We defined the range of $\sec^{-1} x$ to be $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$. An alternative definition takes the range to be $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. Show that with this alternate range, the derivative rule becomes

$$\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}}, \quad \text{for} \quad |x| > 1.$$

$$\sec\left(\sec^{-1} x\right) = x$$

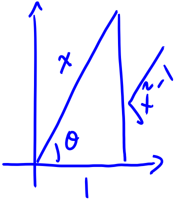
$$\frac{d}{dx}\left[\sec\left(\sec^{-1} x\right)\right] = \frac{d}{dx}[x] = 1$$

$$\sec\left(\sec^{-1} x\right)\tan\left(\sec^{-1} x\right)\frac{d}{dx}\sec^{-1} x = 1$$

Find $\tan(\sec^{-1} x)$

Case: $x > 1$

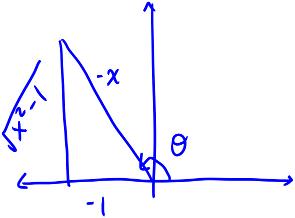
$\theta = \sec^{-1} x$



$\tan(\theta) = \sqrt{x^2-1}$

Case: $x < -1$

$\theta = \sec^{-1} x$



Here $\tan\theta = \dfrac{\sqrt{x^2-1}}{-1} = -\sqrt{x^2-1}$

For $x > 1$, $\quad \frac{d}{dx}\sec^{-1}x = \frac{1}{\sec(\sec^{-1}x)\tan(\sec^{-1}x)}$

$$= \frac{1}{x\sqrt{x^2-1}}$$

For $x < -1$, $\quad \frac{d}{dx}\sec^{-1}x = \frac{1}{x(-\sqrt{x^2-1})} = \frac{1}{-x\sqrt{x^2-1}}$

Which combines to

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}} \quad \text{for } |x| > 1$$