January 15 Math 3260 sec. 51 Spring 2020

Section 1.2: Row Reduction and Echelon Forms

- We defined row echelon (ref) and reduced row echelon (rref) forms.
- We saw that there is an algorithm involving row operations to obtain an (r)ref from a matrix.
- We recall that matrices obtained from elementary row operations are row equivalent. We have the following theorem:

Theorem: The reduced row echelon form of a matrix is unique.

Example

We considered the matrix $\begin{bmatrix} 0 & 3 & -6 & 4 & 6 \\ 3 & -7 & 8 & 8 & -5 \\ 3 & -9 & 12 & 6 & -9 \end{bmatrix}$.

$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ is **an** ref of this matrix.

$\begin{bmatrix} 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ is **THE** rref of this matrix.

Our theorem can be restated as saying: **Every matrix is row equivalent to exactly one reduced echelon matrix.**

Uniqueness of the rref

We have the following unambiguous definitions:

Definition: A **pivot position** in a matrix A is a location that corresponds to a leading 1 in the reduced echelon form of A.

Definition: A **pivot column** is a column of A that contains a pivot position.

We can identify the pivot positions and pivot columns of a matrix by reference to its rref.

Example

Using the know rref to identify the pivot position and columns of the matrix A.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \qquad \text{rref}(A) = \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot positions.

The leading ones are in columns 1, 2, and 4

The 1st, 2nd, and 4th columns of A are the pivot columns

* The pivot positions aren't obvious from A alone. The rref identifies them. *

Complete Row Reduction isn't needed to find Pivots

Pivot positions correspond to leading entries in any ref.

$$\begin{bmatrix} 1 & 1 & 4 \\ -2 & 1 & -2 \\ 1 & 0 & 2 \end{bmatrix}$$

Pivot positions are where leading entries are in any ref.

This matrix has an ref and rref

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{respectively.}$$

It's clear that the first and second columns are pivot columns from the ref.

Solutions to Linear Systems

Example: Suppose the given reduced echelon matrix is an augmented matrix for a linear system. Describe the solution set of the linear system.

$$\begin{array}{c}
\left[\begin{array}{ccccc|c}
1 & 1 & 0 & 0 & 4 & 3\\
0 & 0 & 1 & 0 & -2 & 4\\
0 & 0 & 0 & 1 & 0 & -9\\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]\\
\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}
\end{array}$$

4 equations, 5 variables

The system is

$$\begin{aligned}
x_1 + x_2 \qquad + 4x_5 &= 3\\
x_3 \quad - 2x_5 &= 4\\
x_4 \qquad &= -9
\end{aligned}$$

Clearly, $x_4 = -9$. Otherwise, we can express x_1, x_2, x_3, and x_5 in terms of one another. Note that the pivot columns are 1, 3, and 4. So x_1, x_3, x_4 appear only once with coefficient one.

Columns 2 and 5 (as well as 6) are not pivot columns. Variables x_2, x_5 need not have coefficient one and can appear in more than one equation.

We'll call x_1, x_3, x_4 basic variables, and express them in terms of x_2 and x_5. We'll call x_2 and x_5 free.

We have

$$\left.\begin{aligned} x_1 &= 3 - x_2 - 4x_5 \\ x_3 &= 4 + 2x_5 \\ x_4 &= -9 \end{aligned}\right\} \text{describes the solution set.}$$

and x_2, x_5 are any real number.

* Note: An equation like $x_3 - 2x_5 = 4$
can be written as
$x_3 = 4 + 2x_5$, x_5 any real number OR
$x_5 = -2 + \frac{1}{2}x_3$, x_3 any real number

To make it easier to read and write our results, we'll use the convention that
$x_3 = 4 + 2x_5$ is prefered.

* Column 3 is a pivot column and Column 5 is not

Basic and Free Variables

Suppose we use an augmented matrix and row reduction (to rref) to solve a linear system of equations. If the system is consistent

- We will call a variable a **basic** variable if its column is a pivot column.
- We will call a variable a **free** variable if its column is NOT a pivot column.
- We will follow the convention that when a system has free variables

  We will express basic variables in terms of free variables, and never the other way around.
- Such a description of the solution space is called a **parametric description**.

Example

The equations

$$\begin{array}{lll} x_1 & = & 3 - x_2 - 4x_5 \\ x_3 & = & 4 + 2x_5 \\ x_4 & = & -9 \\ x_2 & \text{and} & x_5 \text{ are free} \end{array}$$

is a **parametric description** of the solution set to the system of equations having $\begin{bmatrix} 1 & 1 & 0 & 0 & 4 & 3 \\ 0 & 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 0 & 1 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ as its augmented matrix.

Consistent versus Inconsistent Systems

Consider each rref. Determine if the underlying system (the one with this as its augmented matrix) is consistent or inconsistent.

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$x_1 + 2x_2 = 0$
$x_3 = 4$
$0 = 0$

$x_1 = -2x_2$
$x_3 = 4$
x_2 - free

Consistent w/ free variable

$x_1 = 0$
$x_2 = 4$
$x_3 = -3$

$x_1 = 0$
$x_2 = 4$
$x_3 = -3$

Consistent no free variables

$x_1 + 2x_3 = 3$
$x_2 + x_3 = 0$
$0 = 1$

$0 = 1$ is false!

inconsistent

An Existence and Uniqueness Theorem

Theorem: A linear system is consistent if and only if the right most column of the augmented matrix is **NOT** a pivot column. That is, if and only if each echelon form **DOES NOT** have a row of the form

$$[0\ 0\ \cdots\ 0\ b], \quad \text{for some nonzero } b.$$

If a linear system is consistent, then it has

(i) exactly one solution if there are no free variables, or
(ii) infinitely many solutions if there is at least one free variable.