Jan. 16 Math 2254H sec 015H Spring 2015

Section 6.6: Inverse Trigonometric Functions

Recall that the inverse sine function is defined by

$$\sin^{-1}(x) = y \quad \text{if and only if} \quad x = \sin y \quad \text{and} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

$$\sin^{-1}(\sin x) = x \quad \text{for} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$\sin\left(\sin^{-1} x\right) = x \quad \text{for} \quad -1 \le x \le 1$$

Arccosine

Similarly

$$\cos^{-1}(x) = y \quad \text{if and only if} \quad x = \cos y \quad \text{and} \quad 0 \le y \le \pi$$

$$\cos^{-1}(\cos x) = x \quad \text{for} \quad 0 \le x \le \pi$$
$$\cos\left(\cos^{-1} x\right) = x \quad \text{for} \quad -1 \le x \le 1$$

Arctangent

$$\tan^{-1}(x) = y \quad \text{if and only if} \quad x = \tan y \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\tan^{-1}(\tan x) = x \quad \text{for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\tan\left(\tan^{-1} x\right) = x \quad \text{for} \quad -\infty < x < \infty$$

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}, \quad \text{and} \quad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

Other Inverse Trig Functions (defined ala Stewart)[1]

$$y = \csc^{-1} x \iff x = \csc y \text{ and } y \in (0, \pi/2] \cup (\pi, 3\pi/2], \ |x| \geq 1$$

$$y = \sec^{-1} x \iff x = \sec y \text{ and } y \in [0, \pi/2) \cup [\pi, 3\pi/2), \ |x| \geq 1$$

$$y = \cot^{-1} x \iff x = \cot y \text{ and } 0 < y < \pi, \ -\infty < x < \infty$$

[1] Unfortunately, different authors may define the range of the inverse secant and cosecant differently. We'll just have to accept this fact. The symbol $\in$ means *"is an element of."*

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}, \qquad \frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}, \qquad \frac{d}{dx}\csc^{-1}x = -\frac{1}{x\sqrt{x^2-1}}$$

Evaluate each derivative

(a) $\frac{d}{dx}\tan^{-1}(3x^3) = \frac{1}{1+(3x^3)^2}\,(9x^2)$

$$= \frac{9x^2}{1+9x^6}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

(b) $\dfrac{d}{dt}\arcsin\left(\dfrac{t}{7}\right) = \dfrac{1}{\sqrt{1-\left(\frac{t}{7}\right)^2}} \cdot \dfrac{1}{7}$

$$= \frac{1}{7\sqrt{1-t^2/7^2}} = \frac{1}{\sqrt{7^2-t^2}}$$

In general $\dfrac{d}{dx}\sin^{-1}\left(\dfrac{x}{a}\right) = \dfrac{1}{\sqrt{a^2-x^2}}$ $a>0$

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

(c) $\frac{d}{dy}\sec^{-1}(\sqrt{y}) = \frac{1}{\sqrt{y}\sqrt{(\sqrt{y})^2-1}} \cdot \frac{1}{2\sqrt{y}}$

$$= \frac{1}{2y\sqrt{y-1}}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$$

Integration Formulas:

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C \qquad \int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C, \quad -a < u < a$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \quad a > 0$$

Derive the Formula for $\int \frac{du}{\sqrt{a^2-u^2}}$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C \quad \text{follows from} \quad \frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}.$$

For $a>0$ and $-a<x<a$, evaluate

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{dx}{\sqrt{a^2(1-x^2/a^2)}} = \int \frac{dx}{|a|\sqrt{1-(x/a)^2}}$$

Let $u = \frac{x}{a}$ $\quad du = \frac{1}{a}\,dx$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$$

$$= \sin^{-1}\left(\frac{x}{a}\right) + C$$

Evaluate each integral

(a) $$\int \frac{x}{\sqrt{1-x^4}}\,dx$$

$\cancel{u = x^4 \qquad du = 4x^3\,dx}$

try again

$$= \int \frac{x}{\sqrt{1^2-(x^2)^2}}\,dx$$

$u = x^2 \qquad du = 2x\,dx$

$\frac{1}{2}\,du = x\,dx$

$$= \frac{1}{2}\int \frac{du}{\sqrt{1^2-u^2}} = \frac{1}{2}\sin^{-1}(u) + C$$

$$= \frac{1}{2}\sin^{-1}(x^2) + C$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

(b) $\displaystyle\int_0^2 \frac{dx}{x^2+4} = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\Big|_0^2$

$$= \frac{1}{2}\tan^{-1}(1) - \frac{1}{2}\tan^{-1}(0)$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right) - 0$$

$$= \frac{\pi}{8}$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

(c) $\displaystyle\int \frac{1}{x^2+2x+2}\,dx$

Complete the square

$$x^2+2x+2 = (x^2+2x+1)+1$$

$$= (x+1)^2+1$$

$$= \int \frac{dx}{(x+1)^2+1}$$

Let $u = x+1$

$du = dx$

$$= \int \frac{du}{u^2+1}$$

$$= \tan^{-1} u + C = \tan^{-1}(x+1) + C$$

(d) $\displaystyle\int \frac{e^{2x}\,dx}{1+e^{4x}}$

$u = e^{2x}$ $\qquad du = 2e^{2x}\,dx$

$\frac{1}{2}\,du = e^{2x}\,dx$

$$= \frac{1}{2}\int \frac{du}{1+u^2}$$

$$= \frac{1}{2}\tan^{-1} u + C$$

$$= \frac{1}{2}\tan^{-1}\left(e^{2x}\right) + C$$

Comparison of Similar (yet different) Integrals

Evaluate $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}$ $= \sin^{-1} x + C$

$u = 1 - x^2$, $du = -2x\,dx$

Evaluate $\displaystyle\int \frac{2x\,dx}{\sqrt{1-x^2}}$

$$= -\int u^{-1/2}\,du = -\frac{u^{1/2}}{1/2} + C = -2\sqrt{u} + C$$

$$= -2\sqrt{1-x^2} + C$$

Recap of Sections 6.2*–6.6

New Differentiation Rules:

$$\frac{d}{dx}\ln x = \frac{1}{x}, \qquad \frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}\ln u = \frac{\frac{du}{dx}}{u}, \qquad \frac{d}{dx}e^u = e^u\frac{du}{dx}$$

$$\frac{d}{dx}\log_a u = \frac{\frac{du}{dx}}{(\ln a)u}, \qquad \frac{d}{dx}a^u = (\ln a)a^u\frac{du}{dx}$$

$$\frac{d}{dx}\sin^{-1} u = \frac{\frac{du}{dx}}{\sqrt{1-u^2}} = -\frac{d}{dx}\cos^{-1} u$$

$$\frac{d}{dx}\tan^{-1} u = \frac{\frac{du}{dx}}{1+u^2} = -\frac{d}{dx}\cot^{-1} u$$

$$\frac{d}{dx}\sec^{-1} u = \frac{\frac{du}{dx}}{u\sqrt{u^2-1}} = -\frac{d}{dx}\csc^{-1} u$$

Examples

(a) $\frac{d}{dx}\exp(\cot^{-1}x) = \exp(\cot^{-1}x)\cdot\frac{-1}{1+x^2}$

$$= \frac{-\exp(\cot^{-1}x)}{1+x^2}$$

(b) $\frac{d}{dr}\left(2^r + \log_2 r\right) = 2^r \ln 2 + \frac{1}{r\ln 2}$

New Integration Rules

$$\int \frac{du}{u} = \ln|u| + C, \qquad \int e^u\, du = e^u + C$$

$$\int a^u\, du = \frac{a^u}{\ln a} + C,$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C,$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C,$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C,$$

For the second formula, we assume $a > 0$ and $a \neq 1$. For the last three, we assume $a > 0$.