January 16 Math 2306 sec. 53 Spring 2019

Section 3: Separation of Variables

Definition: The first order equation $y' = f(x,y)$ is said to be **separable** if the right side has the form

$$f(x,y) = g(x)h(y).$$

That is, a separable equation is one that has the form

$$\frac{dy}{dx} = g(x)h(y).$$

Determine which (if any) of the following are separable.

(c) $\dfrac{dy}{dx} = \sin(xy^2)$

Not separable

(d) $\dfrac{dy}{dt} - te^{t-y} = 0$

$\Rightarrow \dfrac{dy}{dt} = te^{t-y} = t\,e^{t}\,e^{-y}$

It is separable with $g(t) = te^{t}$

$h(y) = e^{-y}$

Solving Separable Equations

Recall that from $\frac{dy}{dx} = g(x)$, we can integrate both sides

$$\int \frac{dy}{dx}\, dx = \int g(x)\, dx.$$

$$\int dy = \int g(x)\, dx$$

$$y = G(x) + C$$ where G is any antiderivative of $g(x)$

This gives a 1-parameter family of solutions.

We'll use this observation!

Solving Separable Equations

Let's assume that it's safe to divide by $h(y)$ and let's set $p(y) = 1/h(y)$. We solve (usually find an implicit solution) by **separating the variables.**

* recall $dy = \frac{dy}{dx}\,dx$

$$\frac{dy}{dx} = g(x)h(y) \quad \Rightarrow \quad \frac{1}{h(y)}\frac{dy}{dx} = g(x) \qquad \text{(divide by } h\text{)}$$

$$p(y)\frac{dy}{dx}\,dx = g(x)\,dx \qquad \text{(multiply by } dx\text{)}$$

$$p(y)\,dy = g(x)\,dx$$

$$\int p(y)\,dy = \int g(x)\,dx$$

$$P(y) = G(x) + C$$ where P and G are antiderivatives of p, and g.

A 1-parameter family of implicit solutions.

Solve the ODE

$$\frac{dy}{dx} = -\frac{x}{y} = -x\left(\frac{1}{y}\right) \Rightarrow y\frac{dy}{dx} = -x \Rightarrow y\,dy = -x\,dx$$

$$\int y\,dy = \int -x\,dx \Rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$$

Let $2C = k$ $\quad y^2 = -x^2 + k$

$$\Rightarrow x^2 + y^2 = k$$

A 1-parameter family of solutions defined implicitly.

An IVP[1]

$$te^{t-y}\,dt - dy = 0, \qquad y(0) = 1$$

In normal form $\qquad te^{t-y}\,dt = dy \Rightarrow \frac{dy}{dt} = te^{t-y}$

$$\frac{dy}{dt} = te^{t}e^{-y}$$

$$\frac{1}{e^{-y}}\,dy = te^{t}\,dt$$

$$\int e^{y}\,dy = \int te^{t}\,dt \qquad u = t \quad du = dt \qquad v = e^{t} \quad dv = e^{t}\,dt$$

$$e^{y} = te^{t} - \int e^{t}\,dt$$

[1] Recall IVP stands for *initial value problem*.

$$e^y = te^t - e^t + C$$

This is a 1-parameter family of solutions to the ODE. We have to apply the condition $y(0)=1$.

Setting $t=0$ and $y=1$

$$e^1 = 0e^0 - e^0 + C \Rightarrow e = 0 - 1 + C \Rightarrow C = e+1$$

The solution to the IVP is given implicitly by

$$\boxed{e^y = te^t - e^t + e + 1}$$

Caveat regarding division by $h(y)$.

Recall that the IVP $\quad \dfrac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0$

has two solutions

$$y(x) = \frac{x^4}{16} \quad \text{and} \quad y(x) = 0.$$

If we separate the variables

$$\frac{1}{\sqrt{y}}\,dy = x\,dx$$

we lose the second solution.

Why? Division by $\sqrt{y}$ assumes $y \neq 0$!

Solutions Defined by Integrals

Recall (Fundamental Theorem of Calculus)

$$\frac{d}{dx}\int_{x_0}^{x} g(t)\,dt = g(x) \quad \text{and} \quad \int_{x_0}^{x} \frac{dy}{dt}\,dt = y(x) - y(x_0).$$

Use this to solve

$$\frac{dy}{dx} = g(x), \quad y(x_0) = y_0$$

$$\frac{dy}{dt} = g(t) \Rightarrow \quad dy = g(t)\,dt$$

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} g(t)\,dt$$

$$y(x) - y(x_0) = \int_{x_0}^{x} g(t)\,dt$$

$$y(x) = y_0 + \int_{x_0}^{x} g(t)\,dt$$

* Note $\quad y(x_0) = y_0 + \int_{x_0}^{x_0} g(t)\,dt = y_0 + 0 = y_0$

and

$$\frac{d}{dx}\, y(x) = \frac{d}{dx}\left(y_0 + \int_{x_0}^{x} g(t)\,dt \right)$$

$$= 0 + \frac{d}{dx}\int_{x_0}^{x} g(t)\,dt = g(x)$$

So $\quad \dfrac{dy}{dx} = g(x).$