January 16 Math 2306 sec. 57 Spring 2018

Section 2: Initial Value Problems

An initial value problem consists of an ODE with additional conditions.

Solve the equation [1]

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}) \tag{1}$$

subject to the *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, y^{(n-1)}(x_0) = y_{n-1}. \tag{2}$$

The problem (1)–(2) is called an *initial value problem* (IVP).

[1] on some interval I containing x_0.

IVPs

First order case:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

1st order

one condition called an initial condition IC

Second order case:

$$\frac{d^2y}{dx^2} = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

If y is position of a moving particle

y_0 - initial position / where it starts

acceleration

y_1 - initial velocity

Example

Given that $y = c_1 x + \frac{c_2}{x}$ is a 2-parameter family of solutions of $x^2 y'' + xy' - y = 0$, solve the IVP

$$x^2 y'' + xy' - y = 0, \quad y(1) = 1, \quad y'(1) = 3$$

$y = c_1 x + c_2 x^{-1}, \quad y' = c_1 - c_2 x^{-2}, \quad y'' = 2c_2 x^{-3}$

$$\begin{aligned} x^2 y'' + xy' - y &= x^2(2c_2 x^{-3}) + x(c_1 - c_2 x^{-2}) - (c_1 x + c_2 x^{-1}) \\ &= 2c_2 x^{-1} + c_1 x - c_2 x^{-1} - c_1 x - c_2 x^{-1} \\ &= x^{-1}(2c_2 - c_2 - c_2) + x(c_1 - c_1) \\ &= x^{-1}(0) + x(0) \quad = 0 \end{aligned}$$

Apply the initial conditions

$$y(1) = c_1 \cdot 1 + \frac{c_2}{1} = 1 \quad \Rightarrow \quad c_1 + c_2 = 1$$

$$y'(1) = c_1 - c_2 (1)^{-2} = 3 \quad \Rightarrow \quad c_1 - c_2 = 3$$

add

$$2c_1 = 4$$

$$c_1 = 2$$

$$c_2 = 1 - c_1 = 1 - 2 = -1$$

$$\boxed{\text{The solution to the IVP is} \quad y = 2x - \frac{1}{x}}$$

Example

Part 1

Show that for any constant c the relation $x^2 + y^2 = c$ is an implicit solution of the ODE

$$\frac{dy}{dx} = -\frac{x}{y}$$

We'll use implicit diff. to show that if y satisfies the relation, it satisfies the ODE.

$$x^2+y^2 = c \implies 2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x \quad \text{for } y \neq 0 \implies \frac{dy}{dx} = \frac{-2x}{2y}$$

$$\implies \boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

Example

Part 2

Use the preceding results to find an explicit solution of the IVP

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(0) = -2$$

condition says that the point $(0,-2)$ is on the graph of y

From the last slide, we know that $x^2+y^2 = C$.

Impose $y(0) = -2$: $0^2 + (-2)^2 = C$

$4 = C$

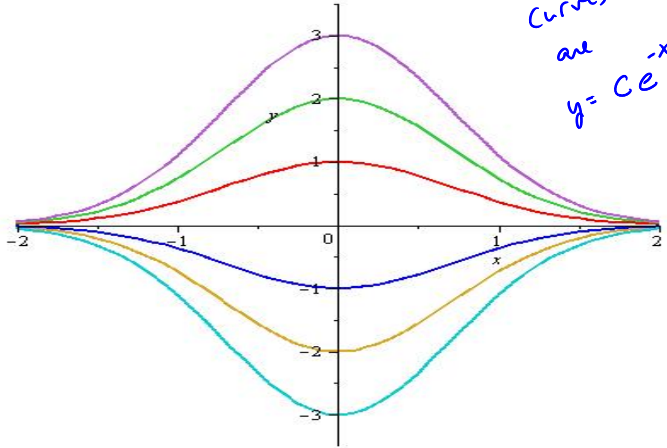
So $x^2+y^2=4$ is an implicit solution of the IVP.

Finding y explicitly,

$$y^2=4-x^2 \quad \Longrightarrow \quad y=\sqrt{4-x^2} \quad \text{or} \quad y=-\sqrt{4-x^2}$$

Since $y(0)=-2$, only the second equation gives the correct solution. The solution to the IVP is $y=-\sqrt{4-x^2}$.

Graphical Interpretation



Curves are $y = Ce^{-x^2}$

Figure: Each curve solves $y' + 2xy = 0$, $y(0) = y_0$. Each colored curve corresponds to a different value of y_0

Example

$x = c_1\cos(2t) + c_2\sin(2t)$ is a 2-parameter family of solutions of the ODE $x'' + 4x = 0$. Find a solution of the IVP

$$x'' + 4x = 0, \quad x\left(\frac{\pi}{2}\right) = -1, \quad x'\left(\frac{\pi}{2}\right) = 4$$

$x' = -2c_1 \sin(2t) + 2c_2 \cos(2t)$

$x(\pi/2) = c_1 \cos\left(2\cdot\frac{\pi}{2}\right) + c_2 \sin\left(2\cdot\frac{\pi}{2}\right) = -1$

$c_1 \cos\pi + c_2 \sin\pi = -1 \implies -c_1 + 0 = -1$

$c_1 = 1$

$x'\left(\frac{\pi}{2}\right) = -2c_1 \sin\left(2\cdot\frac{\pi}{2}\right) + 2c_2 \cos\left(2\cdot\frac{\pi}{2}\right) = 4$

$\implies -2c_2 = 4, \quad c_2 = -2$

The solution to the IVP is

$$x = \cos(2t) - 2\sin(2t).$$

Existence and Uniqueness

Two important questions we can always pose (and sometimes answer) are

(1) Does an IVP have a solution? (existence) and

(2) If it does, is there just one? (uniqueness)

Hopefully it's obvious that we can't solve $\left(\frac{dy}{dx}\right)^2 + 1 = -y^2.$

≥ 1 ≤ 0

Uniqueness

Consider the IVP

$$\frac{dy}{dx} = x\sqrt{y} \quad y(0) = 0$$

Verify that $y = \frac{x^4}{16}$ is a solution of the IVP. And find a second solution of the IVP by clever guessing.

We need to show that $y = \frac{1}{16}x^4$ solves the ODE and the initial condition.

Initial condition: $y(0) = \frac{0^4}{16} = \frac{0}{16} = 0$, yes $y(0) = 0$

ODE: $y = \frac{1}{16}x^4 \Rightarrow \frac{dy}{dx} = \frac{4}{16}x^3 = \frac{1}{4}x^3$

$$x\sqrt{y} = x\sqrt{\frac{1}{16}x^4} = x\,\frac{1}{4}|x^2| = x\left(\frac{x^2}{4}\right) = \frac{x^3}{4}$$

So $\frac{dy}{dx} = \frac{1}{4}x^3 = x\sqrt{y}$ y solves the ODE,

Find a second solution $\frac{dy}{dx} = x\sqrt{y}$, $y(0)=0$

We can look for a constant function $y = C$.

From $y(0)=0$, this requires $C=0$. Does $y=0$ solve the ODE? $y=0 \Rightarrow \frac{dy}{dx} = 0$ and $x\sqrt{y} = x\sqrt{0} = 0$

Yes, so a second solution to the IVP is

$y(x) = 0$. called the trivial solution.