

## Section 2: Initial Value Problems

An initial value problem consists of an ODE with additional conditions.

Solve the equation <sup>1</sup>

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}) \quad (1)$$

subject to the *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, \quad y^{(n-1)}(x_0) = y_{n-1}. \quad (2)$$

The problem (1)–(2) is called an *initial value problem (IVP)*.

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<sup>1</sup>on some interval  $I$  containing  $x_0$ .

# IVPs

First order case:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

*1<sup>st</sup> order ODE*      *↑ one initial condition*      *IC*

Second order case:

$$\frac{d^2y}{dx^2} = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

*2<sup>nd</sup> order ODE*      *2 ICS.*

If  $y$  is the position of a particle @ time  $x$

The ODE gives the acceleration.

$y_0$  would be initial position and

$y_1$  would be the initial velocity.

## Example

Given that  $y = c_1x + \frac{c_2}{x}$  is a 2-parameter family of solutions of  $x^2y'' + xy' - y = 0$ , solve the IVP

$$x^2y'' + xy' - y = 0, \quad y(1) = 1, \quad y'(1) = 3$$

$$y = c_1x + c_2x^{-1}, \quad y' = c_1 - c_2x^{-2}, \quad y'' = 2c_2x^{-3}, \text{ substitute}$$

$$x^2y'' + xy' - y = x^2(2c_2x^{-3}) + x(c_1 - c_2x^{-2}) - (c_1x + c_2x^{-1})$$

$$= 2c_2x^{-1} + c_1x - c_2x^{-1} - c_1x - c_2x^{-1}$$

$$= x^{-1}(2c_2 - c_2 - c_2) + x(c_1 - c_1)$$

$$= x^{-1}(0) + x(0) = 0 \quad \text{as expected}$$

From  $y = c_1 x + \frac{c_2}{x}$ , impose  $y(1) = 1$ ,  $y'(1) = 3$

$$y(1) = c_1(1) + \frac{c_2}{1} = 1 \Rightarrow c_1 + c_2 = 1 \quad \text{2 equations for } c_1, c_2$$

$$y'(1) = c_1 - c_2(1)^{-2} = 3 \Rightarrow c_1 - c_2 = 3$$

$$\underline{2c_1 = 4} \quad \text{add}$$

$$c_1 = 2$$

$$c_2 = 1 - c_1 = 1 - 2 = -1$$

The solution to the IVP

is

$$y = 2x - \frac{1}{x}.$$

## Example

### Part 1

Show that for any constant  $c$  the relation  $x^2 + y^2 = c$  is an implicit solution of the ODE

$$\frac{dy}{dx} = -\frac{x}{y}$$

We use implicit diff. to show that if  $y$  satisfies the relation, it also satisfies the ODE.

$$x^2 + y^2 = c \Rightarrow 2x + 2y \frac{dy}{dx} = 0$$

$$\Rightarrow 2y \frac{dy}{dx} = -2x \quad \text{if } y \neq 0 \quad \frac{dy}{dx} = \frac{-2x}{2y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

## Example

### Part 2

Use the preceding results to find an **explicit** solution of the IVP

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(0) = -2$$

↑ the graph of  $y$  ~~won't~~  
pass through  
 $(0, 2)$ .

We know solutions are given

by  $x^2 + y^2 = C$

Impose  $y(0) = -2$ .

$$0^2 + (-2)^2 = C \Rightarrow 4 = C$$

so implicit solutions to the IVP are

$$x^2 + y^2 = 4.$$

Find  $y$  explicitly:  $y^2 = 4 - x^2$

So

$$y = \sqrt{4 - x^2} \text{ or } y = -\sqrt{4 - x^2}$$

Since  $y(0) = -2$ , only the one with the negative sign solves the IVP.

The solution is  $y = -\sqrt{4 - x^2}$ .

## Graphical Interpretation

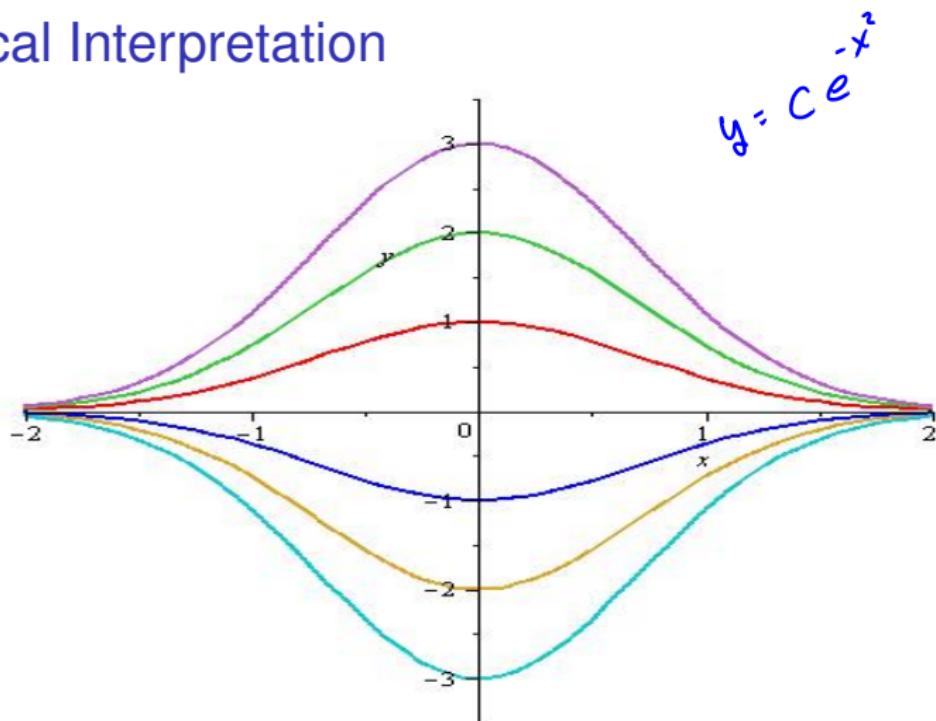


Figure: Each curve solves  $y' + 2xy = 0$ ,  $y(0) = y_0$ . Each colored curve corresponds to a different value of  $y_0$

## Example

$x = c_1 \cos(2t) + c_2 \sin(2t)$  is a 2-parameter family of solutions of the ODE  $x'' + 4x = 0$ . Find a solution of the IVP

$$x'' + 4x = 0, \quad x\left(\frac{\pi}{2}\right) = -1, \quad x'\left(\frac{\pi}{2}\right) = 4$$

$$x' = -2c_1 \sin(2t) + 2c_2 \cos(2t)$$

$$x\left(\frac{\pi}{2}\right) = c_1 \cos\left(2 \cdot \frac{\pi}{2}\right) + c_2 \sin\left(2 \cdot \frac{\pi}{2}\right) = c_1 \cos(\pi) + c_2 \sin(\pi) = -1$$
$$c_1(-1) + c_2(0) = -1 \Rightarrow c_1 = 1$$

$$x'\left(\frac{\pi}{2}\right) = -2c_1 \sin\left(2 \cdot \frac{\pi}{2}\right) + 2c_2 \cos\left(2 \cdot \frac{\pi}{2}\right)$$
$$= -2c_1(0) + 2c_2(-1) = 4 \Rightarrow -2c_2 = 4 \quad c_2 = -2$$

The solution to the IVP is

$$X = \cos(2t) - 2 \sin(2t)$$

# Existence and Uniqueness

Two important questions we can always pose (and sometimes answer) are

- (1) Does an IVP have a solution? (existence) and
- (2) If it does, is there just one? (uniqueness)

Hopefully it's obvious that we can't solve  $\left(\frac{dy}{dx}\right)^2 + 1 = -y^2$ .

*??* *≤ 0*

# Uniqueness

Consider the IVP

$$\frac{dy}{dx} = x\sqrt{y} \quad y(0) = 0$$

Verify that  $y = \frac{x^4}{16}$  is a solution of the IVP. And find a second solution of the IVP by clever guessing.

We need to show that  $y = \frac{1}{16}x^4$  solves both the ODE and the initial condition.

Initial condition:  $y = \frac{1}{16}x^4 \quad y(0) = \frac{1}{16}(0)^4 = 0 \quad y'(0) = 0$

ODE:  $y = \frac{1}{16}x^4 \Rightarrow y' = \frac{1}{16}(4x^3) = \frac{1}{4}x^3$

and  $x\sqrt{y} = x\sqrt{\frac{1}{16}x^4} = x\left(\frac{1}{4}\right)|x^2| = x\left(\frac{1}{4}x^2\right) = \frac{1}{4}x^3$

$$\text{So } \frac{dy}{dx} = \frac{1}{4}x^3 = x\sqrt[4]{y} \quad y = \frac{1}{16}x^4 \text{ solves the ODE.}$$

The IVP is  $y' = x\sqrt[4]{y}$ ,  $y(0) = 0$

If we consider a simple constant function  $y = C$ .

$y(0) = 0$  requires  $C = 0$ ,

For  $y = 0$ ,  $y' = 0$  and  $x\sqrt[4]{y} = x\sqrt[4]{0} = 0$

so  $y = 0$  solves the IVP.

This is called the trivial solution.