January 16 Math 3260 sec. 55 Spring 2018

Section 1.2: Row Reduction and Echelon Forms

We defined the following **Elementary Row Operations**

i Replace a row with the sum of itself and a multiple of another row (**replacement**).

ii Interchange any two rows (**row swap**).

iii Multiply a row by any nonzero constant (**scaling**).

We said that if a sequence of these operations transforms a matrix, the result is called **Row Equivalent.**

Theorem: If the augmented matrices of two linear systems are row equivalent, then the systems are equivalent.

Why is Row Equivalence Interesting?

The following matrices are row equivalent.

$$\begin{bmatrix} 1 & -1 & -1 & 7 \\ 1 & 2 & 2 & 1 \\ \frac{1}{2} & \frac{7}{2} & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Echelon Forms

ref

Definition: A matrix is in **echelon form** (a.k.a. **row echelon form**) if the following properties hold

i. Any row of all zeros are at the bottom.
ii. The first nonzero number (called the *leading entry*) in a row is to the right of the first nonzero number in all rows above it.
iii. All entries below a leading entry are zeros.

Is	Is Not
$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Reduced Echelon Form

rref

Definition: A matrix is in **reduced echelon form** (a.k.a. **reduced row echelon form**) if it is in echelon form and the following additional properties hold

iv The leading entry of each row is 1 (called a *leading* 1), and

v each leading 1 is the only nonzero entry in its column.

Is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Is Not

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Example (finding ref's and rref's)

Find an echelon form for the following matrix using elementary row operations.

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 6 \\ 0 & 3 & 2 \end{bmatrix}$$

We want a nonzero entry in the top left position.

We already have one.

We'll work from left to right, top down.

Considering the 2 in the upper left a leading entry, we use it to clear out the entries below it in it's column.

Do $-2R_1 + R_2 \to R_2$

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 6 \\ 0 & 3 & 2 \end{bmatrix}$$

Scratch

$$\begin{array}{cccc} -4 & -2 & -6 & -2R_1 \\ 4 & 3 & 6 & R_2 \end{array}$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 3 & 2 \end{bmatrix}$$

we want a nonzero entry in row 2 column 2. we already have one.

we consider it a leading entry, and use it to clear the column entries below it.

Do $-3R_2 + R_3 \to R_3$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$0 \quad -3 \quad 0 \quad -3R_2$

$0 \quad 3 \quad 2 \quad R_3$

This is an echelon form.

Find the reduced echelon form for the following matrix.

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 6 \\ 0 & 3 & 2 \end{bmatrix}$$

We have an ref $\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

we work right to left, bottom up.

Scale $\frac{1}{2}R_3 \to R_3$ $\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Clear the 3, do

$-3R_3 + R_1 \to R_1$ $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Clear the 1 in the 2^{nd} column, top row

Do $-R_2 + R_1 \to R_1$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Scale row 1. Do $\frac{1}{2}R_1 \to R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is the rref for the original matrix.

Theorem: The reduced row echelon form of a matrix is unique.

This allows the following unambiguous definition:

Definition: A **pivot position** in a matrix A is a location that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

To identify pivot positions / columns, we can obtain an rref.

Identify the pivot position and columns given...

A

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

rref of A

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

O - pivot positions

The pivot columns are 1, 2, and 4

- leading ones

Complete Row Reduction isn't needed to find Pivots

Find the pivot positions and pivot columns of the matrix

$$\begin{bmatrix} 1 & 1 & 4 \\ -2 & 1 & -2 \\ 1 & 0 & 2 \end{bmatrix}$$

The leading entries in an ref tell us where pivot positions are.

Here the pivot columns are 1 and 2.

This matrix has an ref and rref

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{respectively.}$$

Row Reduction Algorithm

To obtain an echelon form, we work from left to right beginning with the top row working downward.

$$\begin{bmatrix} 0 & 3 & -6 & 4 & 6 \\ 3 & -7 & 8 & 8 & -5 \\ 3 & -9 & 12 & 6 & -9 \end{bmatrix}$$

$(R_1 \leftrightarrow R_3)$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 3 & -7 & 8 & 8 & -5 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Step 1: The left most column is a pivot column. The top position is a pivot position.

Step 2: Get a nonzero entry in the top left position by row swapping if needed.

Row Reduction Algorithm

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 3 & -7 & 8 & 8 & -5 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Do $-R_1 + R_2 \to R_2$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 2 & -4 & 2 & 4 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Step 3: Use row operations to get zeros in all entries below the pivot.

Row Reduction Algorithm

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 2 & -4 & 2 & 4 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

We'll use the 2 to eliminate the 3 in column 2 row 3. Let's scale row 2 first

$\frac{1}{2}R_2 \to R_2$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Do $-3R_2 + R_3 \to R_3$

Step 4: Ignore the row with a pivot, all rows above it, the pivot column, and all columns to its left, and repeat steps 1-3.

Row Reduction Algorithm

$$\begin{array}{rrrrrl} 0 & -3 & 6 & -3 & -6 & -3R_2 \\ 0 & 3 & -6 & 4 & 6 & R_3 \end{array}$$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

This is an echelon form.

Row Reduction Algorithm

To obtain a reduced row echelon form:

Step 5: Starting with the right most pivot and working up and to the left, use row operations to get a zero in each position above a pivot. Scale to make each pivot a 1.

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Use this leading 1 to clear out its column.

$-1R_3 + R_2 \to R_2$ and $-6R_3 + R_1 \to R_1$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Use leading 1 to clear column

Row Reduction Algorithm

Do $9R_2 + R_1 \to R_1$

$$\begin{bmatrix} 3 & 0 & 6 & 6 & 9 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Scale the 1st leading entry

Do $\frac{1}{3}R_1 \to R_1$

$$\begin{bmatrix} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Row Reduction Algorithm

We're done. The rref is

$$\begin{bmatrix} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$