# January 16 Math 3260 sec. 56 Spring 2018

## Section 1.2: Row Reduction and Echelon Forms

We defined the following **Elementary Row Operations**

i Replace a row with the sum of itself and a multiple of another row (**replacement**).

ii Interchange any two rows (**row swap**).

iii Multiply a row by any nonzero constant (**scaling**).

We said that if a sequence of these operations transforms a matrix, the result is called **Row Equivalent.**

**Theorem:** If the augmented matrices of two linear systems are row equivalent, then the systems are equivalent.

# Why is Row Equivalence Interesting?

The following matrices are row equivalent.

$$\begin{bmatrix} 1 & -1 & -1 & 7 \\ 1 & 2 & 2 & 1 \\ \frac{1}{2} & \frac{7}{2} & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

# Echelon Forms

ref

**Definition:** A matrix is in **echelon form** (a.k.a. **row echelon form**) if the following properties hold

i Any row of all zeros are at the bottom.

ii The first nonzero number (called the *leading entry*) in a row is to the right of the first nonzero number in all rows above it.

iii All entries below a leading entry are zeros.

| Is | Is Not |
|---|---|
| $\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ |

# Reduced Echelon Form

rref

**Definition:** A matrix is in **reduced echelon form** (a.k.a. **reduced row echelon form**) if it is in echelon form and the following additional properties hold

iv The leading entry of each row is 1 (called a *leading* 1), and

v each leading 1 is the only nonzero entry in its column.

Is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Is Not

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

# Example (finding ref's and rref's)

Find an echelon form for the following matrix using elementary row operations.

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 6 \\ 0 & 3 & 2 \end{bmatrix}$$

We want a nonzero entry in column 1, row 1. We have one. We'll work from left to right, top-down. We can use the 2 to clear the nonzero entries below it.

To clear the 4, do $-2R_1 + R_2 \to R_2$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 3 & 2 \end{bmatrix}$$

Scratch

$$\begin{array}{cccc} -4 & -2 & -6 & -2R_1 \\ 4 & 3 & 6 & \end{array}$$

We want a nonzero entry in row 2 column 2 if possible. We already have one. We use that leading entry to get zero below it.

Do $-3R_2 + R_3 \to R_3$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

This an echelon form of the original matrix.

Find the reduced echelon form for the following matrix.

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 6 \\ 0 & 3 & 2 \end{bmatrix}$$

From an echelon form, we work right to left, bottom up.

From $\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ Scale Do $\frac{1}{2}R_3 \to R_3$

$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Clear the column $-3R_3 + R_1 \to R_1$

$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Do $-R_2 + R_1 \to R_1$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Finally scale

$\frac{1}{2}R_1 \to R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is the reduced row echelon form of the matrix.

**Theorem:** The reduced row echelon form of a matrix is unique.

This allows the following unambiguous definition:

**Definition:** A **pivot position** in a matrix $A$ is a location that corresponds to a leading 1 in the reduced echelon form of $A$. A **pivot column** is a column of $A$ that contains a pivot position.

# Identify the pivot position and columns given...

$A$

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

O - pivot positions

The pivot columns are 1, 2, and 4.

rref of $A$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

● - leading ones.

# Complete Row Reduction isn't needed to find Pivots

Find the pivot positions and pivot columns of the matrix

$$\begin{bmatrix} 1 & 1 & 4 \\ -2 & 1 & -2 \\ 1 & 0 & 2 \end{bmatrix}$$

The leading entries in any ref are where leading 1s are in the rref.

pivot columns are 1 and 2. The pivot positions are circled.

This matrix has an ref and rref

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{respectively.}$$

# Row Reduction Algorithm

To obtain an echelon form, we work from left to right beginning with the top row working downward.

$$\begin{bmatrix} 0 & 3 & -6 & 4 & 6 \\ 3 & -7 & 8 & 8 & -5 \\ 3 & -9 & 12 & 6 & -9 \end{bmatrix}$$

Get a nonzero entry in the top left position

$(R_1 \leftrightarrow R_3)$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 3 & -7 & 8 & 8 & -5 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Step 1: The left most column is a pivot column. The top position is a pivot position.

Step 2: Get a nonzero entry in the top left position by row swapping if needed.

# Row Reduction Algorithm

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 3 & -7 & 8 & 8 & -5 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Use the 1st row/column entry to clear its column.

$$-R_1 + R_2 \Rightarrow R_2$$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 2 & -4 & 2 & 4 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Step 3: Use row operations to get zeros in all entries below the pivot.

# Row Reduction Algorithm

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 2 & -4 & 2 & 4 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Now, we try to get a nonzero number in row 2 column 2. We have one already. Use it to get a zero below it.

Let's scale row 2 $\quad \frac{1}{2}R_2 \to R_2$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Step 4: Ignore the row with a pivot, all rows above it, the pivot column, and all columns to its left, and repeat steps 1-3.

# Row Reduction Algorithm

$$\begin{array}{lccccc} -3R_2 \to & 0 & -3 & 6 & -3 & -6 \\ R_3 \to & 0 & 3 & -6 & 4 & 6 \end{array}$$

Do $-3R_2 + R_3 \to R_3$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

This is an echelon form.

# Row Reduction Algorithm

To obtain a reduced row echelon form:

Step 5: Starting with the right most pivot and working up and to the left, use row operations to get a zero in each position above a pivot. Scale to make each pivot a 1.

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

use this to get zeros above it

$-R_3 + R_2 \to R_2$ and $-6R_3 + R_1 \to R_1$

$$\begin{bmatrix} 3 & -9 & 12 & 0 & -9 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

use leading one to clear column

# Row Reduction Algorithm

$$9R_2 + R_1 \to R_1$$

$$\begin{bmatrix} 3 & 0 & 6 & 6 & 9 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Finally, scale row 1

$$\frac{1}{3}R_1 \to R_1$$

$$\begin{bmatrix} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

# Row Reduction Algorithm

The rref is

$$\begin{bmatrix} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

we're done!