January 16 Math 3260 sec. 56 Spring 2018

Section 1.2: Row Reduction and Echelon Forms

We defined the following Elementary Row Operations

- i Replace a row with the sum of itself and a multiple of another row (**replacement**).
- ii Interchange any two rows (row swap).
- iii Multiply a row by any nonzero constant (scaling).

We said that if a sequence of these operations transforms a matrix, the result is called **Row Equivalent**.

Theorem: If the augmented matrices of two linear systems are row equivalent, then the systems are equivalent.

Why is Row Equivalence Interesting?

The following matrices are row equivalent.

$$\begin{bmatrix} 1 & -1 & -1 & 7 \\ 1 & 2 & 2 & 1 \\ \frac{1}{2} & \frac{7}{2} & 1 & 3 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

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Echelon Forms

ref

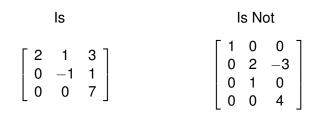
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Definition: A matrix is in **echelon form** (a.k.a. **row echelon form**) if the following properties hold

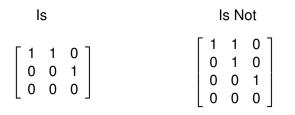
- i Any row of all zeros are at the bottom.
- ii The first nonzero number (called the *leading entry*) in a row is to the right of the first nonzero number in all rows above it.
- iii All entries below a leading entry are zeros.



Reduced Echelon Form

Definition: A matrix is in **reduced echelon form** (a.k.a. **reduced row echelon form**) if it is in echelon form and the following additional properties hold

- iv The leading entry of each row is 1 (called a *leading* 1), and
- v each leading 1 is the only nonzero entry in its column.



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Example (finding ref's and rref's)

Find an echelon form for the following matrix using elementary row operations.

[2 1 3] [4 3 6] [0 3 2] We have one. We'll work from lift to right, top-down. We can use the Z to dear the ronzero entries blow it.

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 3 & z \end{bmatrix}$$
 Scratd,
-4 -2 -6 -2R,
4 3 6

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We want a nonzero entry in row 2 colum 2 if possible, we already have one, we use that leading entry to get zero below it. Do -3R2 + R3 + R3

2 i 3
0 i 0
6 0 2This an echelon form of
the original natrix.

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Find the reduced echelon form for the following matrix.

2 1 3 4 3 6 0 3 2 From an echelon form, we work right to left, bottom up. Scale Do Zk3+R3 $F_{nn} \left(\begin{array}{ccc} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{array} \right)$ $\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Clear the column -3R3 + R, \Rightarrow R, 0 0 1] $\mathcal{D}_{0} = \mathcal{R}_{2} + \mathcal{R}_{1} \Rightarrow \mathcal{R}_{1}$

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$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
 Finally scale
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is the reduced too echelon form of
the motrix.

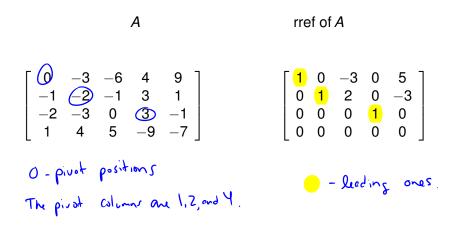
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Theorem: The reduced row echelon form of a matrix is unique.

This allows the following unambiguous definition:

Definition: A **pivot position** in a matrix *A* is a location that corresponds to a leading 1 in the reduced echelon form of *A*. A **pivot column** is a column of *A* that contains a pivot position.

Identify the pivot position and columns given...



Complete Row Reduction isn't needed to find Pivots Find the pivot positions and pivot columns of the matrix

$$\begin{bmatrix} 1 & 1 & 4 \\ -2 & 1 & -2 \\ 1 & 0 & 2 \end{bmatrix}$$
 The leading entriver in any ret are where leading 1s are in the met.
pivot columns are land 2. The pivot positions are circled.

This matrix has an ref and rref

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \text{ respectively.}$$

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To obtain an echelon form, we work from left to right beginning with the top row working downward.

$$\begin{bmatrix} 0 & 3 & -6 & 4 & 6 \\ 3 & -7 & 8 & 8 & -5 \\ 3 & -9 & 12 & 6 & -9 \end{bmatrix} \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \text{ left position} \\ k_1 \leftrightarrow k_2 \\ R_3 \rightarrow R_3 \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \text{ left position} \\ k_2 \rightarrow R_3 \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \leftrightarrow R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \to R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \to R_3) \xrightarrow{\text{Get a nonzero entry in the top}} (R_1 \to R_3) \xrightarrow{\text{Get a nonzero entry in top}} (R_1 \to R_3) \xrightarrow{\text{Get a nonzero entry in top}} (R_1 \to R_3) \xrightarrow$$

Step 1: The left most column is a pivot column. The top position is a pivot position. Step 2: Get a nonzero entry in the top left position by row swapping if needed.

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 3 & -7 & 8 & 8 & -5 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$
Use the 1st row claim entry to
Clear its Column.
- $R_1 + R_2 \Rightarrow R_2$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 2 & -4 & 2 & 4 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Step 3: Use row operations to get zeros in all entries below the pivot.

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$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 2 & -4 & 2 & 4 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix} \xrightarrow{\text{nonzero number in row 2}}_{\text{column 2. We have one}} \xrightarrow{\text{onzero number in row 2}}_{\text{column 2. We have one}}_{\text{column 2. We have one}}_{$$

Step 4: Ignore the row with a pivot, all rows above it, the pivot column, and all columns to its left, and repeat steps 1-3.

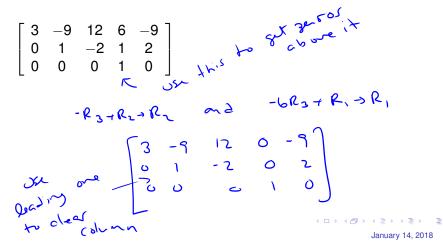
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 $-3R_{2} - 3 - 3 - 3 - 3 - 3 - 6$ Row Reduction Algorithm $f_{3} > 0 - 3 - 6 - 3 - 6 - 4 - 6$ $D_{0} - 3R_{2} + R_{3} - 3R_{3}$ $\begin{bmatrix} 3 - 9 & 12 - 6 - 9 \\ 0 - 1 - 2 - 1 - 2 \\ 0 - 0 - 3 - 1 - 0 \end{bmatrix}$

This is an echelon form.

To obtain a reduced row echelon form:

Step 5: Starting with the right most pivot and working up and to the left, use row operations to get a zero in each position above a pivot. Scale to make each pivot a 1.



$$\begin{bmatrix}
1 & 0 & 2 & 2 & 3 \\
0 & 1 & -2 & 0 & 2 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}$$

Were done

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