January 17 MATH 1112 sec. 54 Spring 2020

Linear Functions and Linear Equations

Let m and b be real numbers with $m \neq 0$. A function of the form

- $f(x) = b$ is called a constant function, and
- $f(x) = mx + b$ is called a linear function.

The value of m in a linear function is the *slope* of the linear function. If $y = f(x)$, the slope is a measure of the rate of change of the dependent variable y given a change in the independent variable x. For a specified change in x, say Δx

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$

Average Rate of Change

Many interesting functions are not linear. However, we often use linear approximations to analyze the behavior of functions over small intervals.

Definition: Suppose $x_1 < x_2$, and that a function f is defined over an interval $[x_1, x_2]$. Then the **average rate of change** of f on the interval $[x_1, x_2]$ is

$$\text{average rate of change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Notice the slope formula there! The average rate of change is a slope. It's the slope of the line connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on the graph of f.

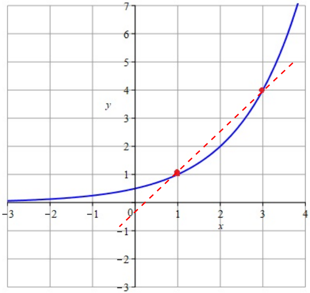
Example

ARC "average rate of change"

The graph of $y = f(x)$ passes through the points $(1, 1)$ and $(3, 4)$ as seen in the figure. Find the average rate of change of f over the interval $[1, 3]$.

$(x_1, f(x_1)) = (1, 1)$

$(x_2, f(x_2)) = (3, 4)$



$$ARC = \frac{4-1}{3-1} = \frac{3}{2}$$

* Note, this is the slope of the line through these points on the curve.

The line is called a <u>secant</u> line.

Two Linear Equations in Two Variables

Jazz empties his pocket to find 21 coins, all nickels and pennies. He counts 49 cents. How many nickels does he have?

If we let P be the number of pennies and N the number of nickels, the above scenario can be expressed mathematically by the pair of equations

$$\begin{array}{rcrcll} P & + & N & = & 21 & \text{(number of coins equation)} \\ 1P & + & 5N & = & 49 & \text{(number of cents equation)} \end{array}$$

Both equations are supposed to hold true, so the numbers P and N must satisfy this **system of equations**.

Linear System

$$\begin{array}{rcrcl} P & + & N & = & 21 \\ P & + & 5N & = & 49 \end{array}$$

Observe that we can rewrite these equations (in a couple of different ways) such as

$$N = -P + 21 \quad \text{and} \quad N = -\frac{1}{5}P + \frac{49}{5}.$$

Noting the resemblance to the slope intercept form of a **line**

$$y = mx + b$$

we see why this is called *linear system*.

Example Solving a System

$$\begin{aligned} P &+ N &= 21 \\ P &+ 5N &= 49 \end{aligned}$$

we have $P = 21 - N$ and

$P = 49 - 5N.$

Since $P = P$

$21 - N = 49 - 5N$

$\Rightarrow \quad 4N = 28$

$N = 7$

Then substituting $P = 21 - N = 21 - 7 = 14$

So Jazz has 14 pennies and 7 nickels

Quick verification:

$$P + N = 21 \quad ?$$

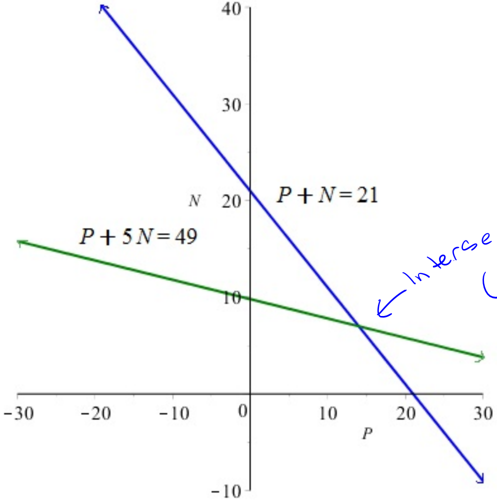
$$14 + 7 = 21 \quad \checkmark$$

$$P + 5N = 49$$

$$14 + 5(7) = 14 + 35 = 49 \quad \checkmark$$

Yep!

System Solution: Graphical Representation



← Intersection (14, 7) that's (P, N)

Figure: The lines $N = -P + 21$ and $N = -\frac{1}{5}P + \frac{49}{5}$ on one set of axes.

Two Equations in Two Variables

Theorem: Let a, b, c, d, e, and f be fixed constants. The system of equations

$$\begin{array}{ccccc} ax & + & by & = & e \\ cx & + & dy & = & f \end{array}$$

satisfies one of three cases:

- It has exactly one solution.
- It has infinitely many solutions.
- It has no solutions.

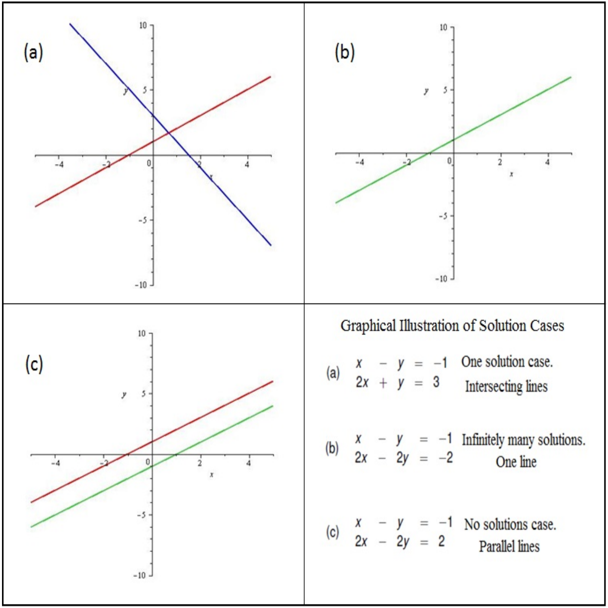
If the system has a solution (first two cases), it is called **consistent**. If it has no solutions, it is called **inconsistent**.

Consistent and Inconsistent Systems

Consistent Independent: A system is called this when it has exactly one solution. Graphically, two lines intersect in one point.

Consistent Dependent: A system is called this when it has infinitely many solutions. Graphically, the equations define the same line. All points on that line represent solutions.

Inconsistent: A system is called this when it has no solutions. Graphically, the equations define distinct, parallel lines.


(a)
(b)
(c)
Graphical Illustration of Solution Cases
(a) x − y = −1 One solution case.
2x + y = 3 Intersecting lines
(b) x − y = −1 Infinitely many solutions.
2x − 2y = −2 One line
(c) x − y = −1 No solutions case.
2x − 2y = 2 Parallel lines

Example

Determine if the system is consistent. If so, characterize the solution.

$$\begin{array}{rcrcl} 2x & + & y & = & 3 \\ x & + & \frac{1}{2}y & = & \frac{3}{2} \end{array}$$

The 2^{nd} equation gives $x = \frac{3}{2} - \frac{1}{2}y$.

Sub into the 1^{st}

$$2\left(\frac{3}{2} - \frac{1}{2}y\right) + y = 3$$

$$3 - y + y = 3$$

$$3 = 3$$

This equation is always true.

* one line case

There are infinitely many solutions. Both equations define the same line.

All points on the line are solutions.

We can express the solutions in set builder notation as

$$\{(x,y) \mid y = -2x+3\}$$

* Note the 1st equation $2x+y=3$ is equivalent to $y=-2x+3$ *

The system is consistent dependent.

Example

Determine if the system is consistent. If so, characterize the solution.

$$\begin{array}{rcrcr} 2x & + & y & = & 3 \\ x & + & \frac{1}{2}y & = & -4 \end{array}$$

The 2^{nd} equation gives $x = -4 - \frac{1}{2}y$.

Sub into the 1^{st}

$$2\left(-4 - \tfrac{1}{2}y\right) + y = 3$$

$$-8 - y + y = 3$$

$$-8 = 3$$ huh?

This is always false! * Parallel line case

We obtain a statement that is never true.

This tells us that the system is inconsistent.

The solution set is empty.