January 17 Math 3260 sec. 55 Spring 2020

Section 1.3: Vector Equations

Definition: A matrix that consists of one column is called a **column vector** or simply a **vector**.

Denoting Vectors:

- Bold faced in typesetting: vector $\mathbf{x}$ and number x
- Arrow overscore in handwriting: vector $\vec{x}$ and number x.

$\vec{x}$ versus x

vector number

$\mathbb{R}^2$ & Geometry

The set of vectors of the form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with x_1 and x_2 real numbers is denoted by $\mathbb{R}^2$ (read "R two"). It's the set of all real ordered pairs.

Each vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ corresponds to a point in the Cartesian plane. We can equate them with ordered pairs written in the traditional format $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (x_1, x_2)$. This is **not to be confused with a row matrix.**

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq [x_1 \;\; x_2]$$

no comma here

We can identify vectors with points or with directed line segments emanating from the origin (little arrows).

Geometry

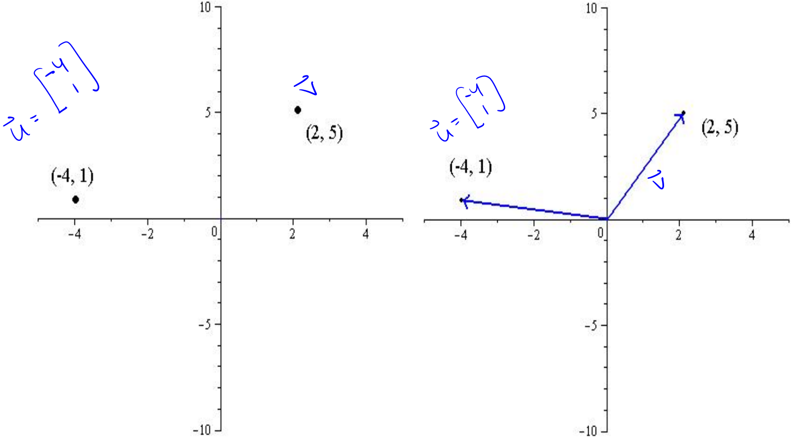


Figure: Vectors characterized as points, and vectors characterized as directed line segments.

Algebraic Operations

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, and c be a scalar[1].

Scalar Multiplication: The scalar multiple of $\mathbf{u}$

$$c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}.$$

Vector Addition: The sum of vectors $\mathbf{u}$ and $\mathbf{v}$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

Vector Equivalence: Equality of vectors is defined by

$$\mathbf{u} = \mathbf{v} \quad \text{if and only if} \quad u_1 = v_1 \quad \text{and} \quad u_2 = v_2.$$

[1] A **scalar** is an element of the set from which u_1 and u_2 come. For our purposes, a scalar is a *real* number.

Examples

$$\mathbf{u} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} -3 \\ \frac{3}{2} \end{bmatrix}$$

Evaluate

(a) $-2\mathbf{u}$ $= -2 \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -2(4) \\ -2(-2) \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$

$3\vec{v} = \begin{bmatrix} 3(-1) \\ 3(7) \end{bmatrix} = \begin{bmatrix} -3 \\ 21 \end{bmatrix}$

(b) $-2\mathbf{u}+3\mathbf{v}$ $= \begin{bmatrix} -8 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ 21 \end{bmatrix} = \begin{bmatrix} -8-3 \\ 4+21 \end{bmatrix} = \begin{bmatrix} -11 \\ 25 \end{bmatrix}$

Is it true that $\mathbf{w} = -\frac{3}{4}\mathbf{u}$? well, $\frac{-3}{4}\vec{u} = \frac{-3}{4}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ \frac{3}{2} \end{bmatrix}$

as both components match, yes

$\vec{w} = \frac{-3}{4}\vec{u}$.

Geometry of Algebra with Vectors

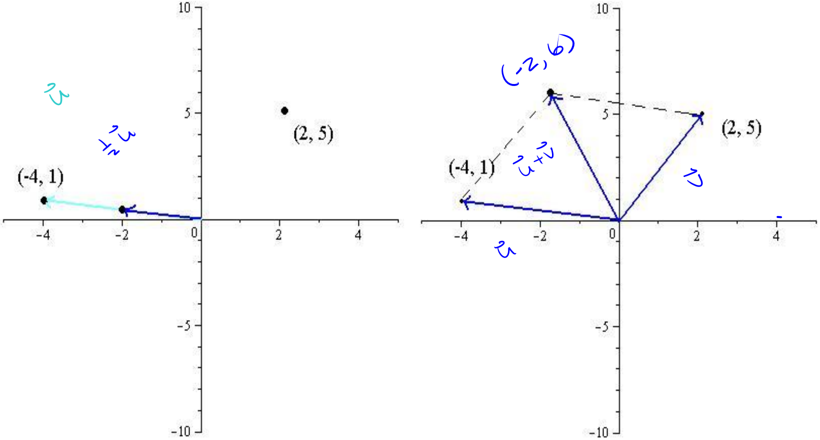
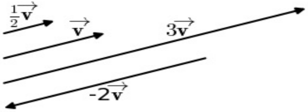
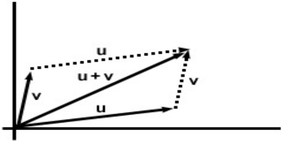


Figure: Left: $\frac{1}{2}(-4, 1) = (-2, 1/2)$. Right: $(-4, 1) + (2, 5) = (-2, 6)$

Geometry of Algebra with Vectors

Scalar Multiplication: stretches or compresses a vector but can only change direction by an angle of 0 (if $c > 0$) or π (if $c < 0$). We'll see that $0\mathbf{u} = (0,0)$ for any vector $\mathbf{u}$ in $\mathbb{R}^2$.





Vector Addition: The sum $\mathbf{u} + \mathbf{v}$ of two vectors (nonparallel and not $(0,0)$) is the the fourth vertex of a parallelogram whose other three vertices are (u_1, u_2), (v_1, v_2), and $(0,0)$.

Vectors in $\mathbb{R}^n$

A vector in $\mathbb{R}^3$ is a 3×1 column matrix. These are ordered triples. For example

$$\mathbf{a} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \quad \text{or} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

A vector in $\mathbb{R}^n$ for $n \geq 2$ is a $n \times 1$ column matrix. These are ordered n-tuples. For example

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The Zero Vector: is the vector whose entries are all zeros. It will be denoted by $\mathbf{0}$ or $\vec{0}$ and is not to be confused with the scalar 0.

Algebraic Properties on $\mathbb{R}^n$

For every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^n$ and scalars c and d[2]

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(iv) $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$

(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(vii) $c(d\mathbf{u})=d(c\mathbf{u})=(cd)\mathbf{u}$

(viii) $1\mathbf{u}=\mathbf{u}$

These all follow fairly easily from our definitions. We'll note the structure. We'll see this structure again later!

[2] The term $-\mathbf{u}$ denotes $(-1)\mathbf{u}$.

Definition: Linear Combination

A linear combination of vectors $\mathbf{v}_1, \dots \mathbf{v}_p$ in $\mathbb{R}^n$ is a vector $\mathbf{y}$ of the form

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

where the scalars $c_1, \dots, c_p$ are often called weights.

For example, suppose we have two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Some linear combinations include

$$3\mathbf{v}_1, \quad -2\mathbf{v}_1 + 4\mathbf{v}_2, \quad \frac{1}{3}\mathbf{v}_2 + \sqrt{2}\mathbf{v}_1, \quad \text{and} \quad \mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2.$$

Note $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2$

zero vector

zero number

Example

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$. Determine if $\mathbf{b}$ can be written as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

Do there exist numbers c_1, c_2 such that

$$c_1\vec{a}_1 + c_2\vec{a}_2 = \vec{b}\ ?$$

$$c_1\vec{a}_1 + c_2\vec{a}_2 = c_1\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 + 3c_2 \\ -2c_1 \\ -c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$$

definition of scalar mult and vector add.

Vector equality requires

$$\begin{aligned} c_1 + 3c_2 &= -2 \\ -2c_1 &= -2 \\ -c_1 + 2c_2 &= -3 \end{aligned}$$

a linear system in variables c_1 and c_2!

From E_2, $c_1 = 1$. Sub into E_1 and E_3

$$3c_2 = -2 - 1 = -3 \Rightarrow c_2 = -1$$
$$2c_2 = -3 + 1 = -2 \Rightarrow c_2 = -1$$

we can solve the system to get $c_1 = 1$, $c_2 = -1$

* Yes $\vec{b}$ is a linear combination of $\vec{a}_1$ and $\vec{a}_2$.

In fact, $\vec{b} = \vec{a}_1 - \vec{a}_2$.

* Note! From the system, we have augmented matrix

$$\begin{bmatrix} 1 & 3 & -2 \\ -2 & 0 & -2 \\ -1 & 2 & -3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

not a pivot column

The system is consistent. Again, we get $c_1 = 1$, $c_2 = -1$.