# January 18 MATH 1112 sec. 54 Spring 2019

## Section 2.2: The Algebra of Functions

Let $f$ and $g$ be functions, and suppose that $x$ is in the domain of each. Then define $f+g$, $f-g$, $fg$ and $f/g$, and use the following notation

- $(f+g)(x) = f(x) + g(x)$
- $(f-g)(x) = f(x) - g(x)$
- $(fg)(x) = f(x)g(x)$
- $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ provided $g(x) \neq 0$

The domain of $f+g$, $f-g$, and $fg$ is the set of all $x$ such that $x$ is in the domain of $f$ and in the domain of $g$. The domain of $f/g$ is the set of all all $x$ such that $x$ is in the domain of $f$, $x$ is in the domain of $g$, and $g(x) \neq 0$.

# Example

Let $f(x) = \dfrac{x}{x+2}$ and $g(x) = 4x - 1$. Find each of

(a) $(fg)(x) = f(x)\,g(x) = \left(\dfrac{x}{x+2}\right)(4x-1) = \dfrac{x(4x-1)}{x+2} = \dfrac{4x^2-x}{x+2}$

(b) $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{\frac{x}{x+2}}{4x-1} = \dfrac{x}{x+2}\left(\dfrac{1}{4x-1}\right)$

$$= \frac{x}{(x+2)(4x-1)} = \frac{x}{4x^2+7x-2}$$

# Question

Let $f(x) = 2x^2$ and $g(x) = \dfrac{2}{x-5}$. Evaluate $(fg)(2)$ and $\left(\dfrac{f}{g}\right)(1)$.

(a) $(fg)(2) = -3$ and $\left(\dfrac{f}{g}\right)(1) = -4$

(b) $(fg)(2) = -\frac{16}{3}$ and $\left(\dfrac{f}{g}\right)(1) = -1$

(c) $(fg)(2) = -\frac{16}{3}$ and $\left(\dfrac{f}{g}\right)(1) = -4$

(d) $(fg)(2) = \frac{16}{3}$ and $\left(\dfrac{f}{g}\right)(1) = -1$

$(fg)(2) = f(2)g(2)$

$= 2(2^2) \cdot \frac{2}{2-5} = 8\left(\frac{-2}{3}\right)$

$= \frac{-16}{3}$

$\left(\frac{f}{g}\right)(1) = \frac{f(1)}{g(1)} = \frac{2(1)^2}{\frac{2}{1-5}}$

$= \frac{2}{\frac{2}{-4}} = -4$

# Average Rate of Change & Difference Quotients

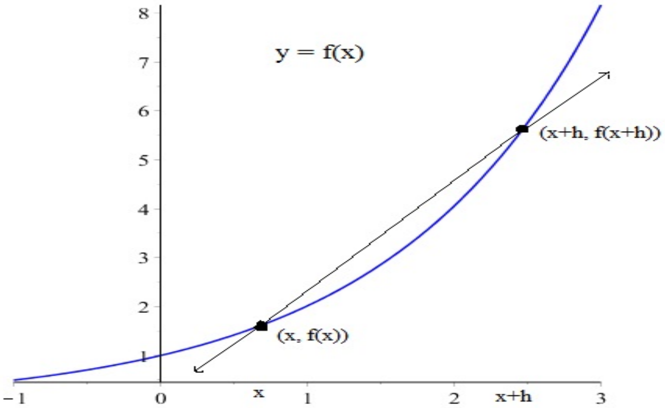


Figure: Consider points $(x, f(x))$ and $(x+h, f(x+h))$ on the graph of $f$ and the straight line through them. This line is called a **secant** line.

# Average Rate of Change: Difference Quotients

Let $f$ be defined at $x$ and $x+h$ for nonzero number $h$. The line passing through $(x, f(x))$ and $(x+h, f(x+h))$ is called a **secant** line. Determine its slope.

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{x+h - x} = \frac{f(x+h) - f(x)}{h}$$

Let $(x, f(x)) = (x_1, y_1)$ and

$(x+h, f(x+h)) = (x_2, y_2)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The result is called a **difference quotient**. It is the **average rate of change** of $f$ on the interval $[x, x+h]$.

# Example

For $h \neq 0$, construct and simplify the difference quotient $\frac{f(x+h)-f(x)}{h}$ for $f(x) = \frac{1}{x}$.

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

Clear the fractions multiply by

$$\frac{x(x+h)}{x(x+h)}$$

$$= \left(\frac{\frac{1}{x+h} - \frac{1}{x}}{h}\right)\frac{x(x+h)}{x(x+h)}$$

$$= \frac{\frac{1}{x+h}(x(x+h)) - \frac{1}{x}(x(x+h))}{hx(x+h)}$$

$$= \frac{x - (x+h)}{hx(x+h)}$$

$$= \frac{x - x - h}{hx(x+h)}$$

$$= \frac{-h}{hx(x+h)}$$

$$= \frac{-1}{x(x+h)} = \frac{f(x+h)-f(x)}{h} \quad \text{for } f(x) = \frac{1}{x}$$

# Question

The difference quotient $\frac{f(x+h)-f(x)}{h}$ for $f(x) = 3x^2 - x$ is

(a) $6x + 3h - 1$

(b) $3h + 1$

(c) $\dfrac{3h^2 - 2x + h}{h}$

(d) $6x - 1$

(e) none of the above is the correct answer

# Section 2.3: Compositions

Suppose a spherical balloon is inflated so that the radius after time $t$ seconds is given by the function $r(t) = 2t$ cm. The volume of a sphere of radius $r$ is known to be $V(r) = \frac{4}{3}\pi r^3$. Note that

- $r$ is a function of $t$, and
- $V$ is a function of $r$, making
- $V$ a function of $t$ (through its dependence on $r$). In fact,

$$V(t) = V(r(t)) = \frac{4}{3}\pi(2t)^3 = \frac{32}{3}\pi t^3.$$

This is an example of a **composition** of functions.

# Composition: Definition and Notation

Let $f$ and $g$ be functions. Then the **composite** function denoted

$$f \circ g,$$

also called the **composition** of $f$ and $g$, is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all $x$ in the domain of $g$ such that $g(x)$ is in the domain of $f$.

The expression $f \circ g$ is read "$f$ composed with $g$", and $(f \circ g)(x)$ is read "$f$ of $g$ of $x$".

# Example

Let $f(x) = \sqrt{x-1}$ and $g(x) = \frac{2}{x+1}$. Evaluate each expression if possible.

(a) $(f \circ g)(1) = f(g(1)) = f\left(\frac{2}{1+1}\right) = f(1) = \sqrt{1-1} = 0$

(b) $(g \circ f)(1) = g(f(1)) = g(\sqrt{1-1}) = g(0) = \frac{2}{0+1} = 2$

(c) $(f \circ g)(0) = f(g(0)) = f\left(\frac{2}{0+1}\right) = f(2) = \sqrt{2-1} = \sqrt{1} = 1$

## Example $f(x) = \sqrt{x - 1}$ and $g(x) = \frac{2}{x+1}$

Find and simplify the function $F(x) = (f \circ g)(x)$.

$$F(x) = f(g(x)) = f\left(\frac{2}{x+1}\right) = \sqrt{\frac{2}{x+1} - 1}$$

$$= \sqrt{\frac{2}{x+1} - \left(\frac{x+1}{x+1}\right)} = \sqrt{\frac{2-(x+1)}{x+1}}$$

$$= \sqrt{\frac{2-x-1}{x+1}} = \sqrt{\frac{1-x}{x+1}}$$