January 18 Math 2306 sec. 54 Spring 2019

Section 3: Separation of Variables

Solutions Defined by Integrals: The separable IVP

$$\frac{dy}{dx} = g(x), \quad y(x_0) = y_0$$

has solution

$$y = y_0 + \int_{x_0}^{x} g(t)\, dt$$

Example

Express the solution of the IVP in terms of an integral.

$$\frac{dy}{dx} = \sin(x^2), \quad y(\sqrt{\pi}) = 1$$

The solution to the IVP is

$$y = 1 + \int_{\sqrt{\pi}}^{x} \sin(t^2)\, dt$$

Section 4: First Order Equations: Linear

A first order linear equation has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$ the equation is called **homogeneous.** Otherwise it is called **nonhomogeneous.**

Provided $a_1(x) \neq 0$ on the interval I of definition of a solution, we can write the **standard form** of the equation

$P(x) = \frac{a_0(x)}{a_1(x)}$

$$\frac{dy}{dx} + P(x)y = f(x).$$

$f(x) = \frac{g(x)}{a_1(x)}$

We'll be interested in equations (and intervals I) for which P and f are continuous on I.

Solutions (the General Solution)

$$\frac{dy}{dx} + P(x)y = f(x).$$

It turns out the solution will always have a basic form of $y = y_c + y_p$ where

- y_c is called the **complementary** solution and would solve the equation

  $$\frac{dy}{dx} + P(x)y = 0$$

  (called the associated homogeneous equation), and
- y_p is called the **particular** solution, and is heavily influenced by the function $f(x)$.

The cool thing is that our solution method will get both parts in one process—we won't get this benefit with higher order equations!

Motivating Example

$$x^2\frac{dy}{dx}+2xy=e^x$$

The equation is not in standard form due to the coefficient x^2.

The interval of definition doesn't contain zero, we can assume that $x>0$.

Note that the left side is $\frac{d}{dx}\left(x^2y\right)$.

So the ODE is

$$\frac{d}{dx}\left(x^2y\right)=e^x$$

Our goal is to find y as a function of x.

Integrate both sides

$$\int \frac{d}{dx}\left(x^2 y\right) dx = \int e^x \, dx$$

$$x^2 y = e^x + C$$

$$\boxed{\text{The solutions are } \quad y = \frac{e^x + C}{x^2}\,.}$$

Note

$$y = \underbrace{\frac{C}{x^2}}_{y_c} + \underbrace{\frac{e^x}{x^2}}_{y_p}$$

Derivation of Solution via Integrating Factor

Solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

We want to create a product rule on the left side.

We'll multiply both sides by a positive function $\mu(x)$ so the left side becomes the derivative of a product.

Multiply by μ $\quad \mu\frac{dy}{dx} + \mu P(x)y = \mu f(x)$

We want this to be

$$\frac{d}{dx}(\mu y) = \mu \frac{dy}{dx} + \frac{d\mu}{dx} y$$

Matching requires

$$\frac{d\mu}{dx} y = \mu P(x) y$$

Cancel y

$$\frac{d\mu}{dx} = \mu P(x)$$

a separable ODE for μ

Separate the variables

$$\frac{1}{\mu}\frac{d\mu}{dx} = P(x)$$

$$\int \frac{1}{\mu}\, d\mu = \int P(x)\, dx$$

$$\ln \mu = \int P(x)\, dx$$

$$\mu = e^{\int P(x)\, dx}$$

μ is called an integrating factor.

The ODE becomes

$$\frac{d}{dx}(\mu y) = \mu(x) f(x)$$

$$\int \frac{d}{dx}(\mu y)\, dx = \int \mu(x) f(x)\, dx$$

$$\mu y = \int \mu(x) f(x)\, dx$$

$$y = \frac{1}{\mu}\left(\int \mu(x) f(x)\, dx\right)$$

General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx} f(x)\,dx + C\right)$$