January 19 Math 2335 sec 51 Spring 2016

Section 1.1: The Taylor Polynomial

Suppose f has at least n continuous derivatives on the interval (α, β) and that a is a point in this interval. The Taylor polynomial of degree n centered at a for the function f is

$$p_n(x) = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

In general, for $x \approx a$ $\quad p_n(x) \approx f(x)$

Example

Use appropriate Taylor polynomials of degree 1 and 2 to approximate $\sqrt{4.1}$.

We need a function and a center.

As we're trying to approximate a square root, we're motivated to choose $f(x)=\sqrt{x}$.

To get a center "close to" 4.1, we can take $a=4$.

$f(x)=\sqrt{x}=x^{1/2}$

$f'(x)=\frac{1}{2}x^{-1/2}$

$f''(x)=\frac{-1}{4}x^{-3/2}$

$f(4)=\sqrt{4}=2$

$f'(4)=\frac{1}{2}(4)^{-1/2}=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$

$f''(4)=\frac{-1}{4}(4)^{-3/2}=\frac{-1}{4}\cdot\frac{1}{8}=\frac{-1}{32}$

$$P_1(x) = f(4) + f'(4)(x-4)$$

$$\boxed{P_1(x) = 2 + \frac{1}{4}(x-4)}$$

$$P_2(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2!}(x-4)^2$$

$$P_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1/32}{2}(x-4)^2$$

$$\boxed{P_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2}$$

$$\sqrt{4.1} = f(4.1) \approx p_1(4.1) = 2 + \frac{1}{4}(4.1-4)$$

$$= 2 + \frac{1}{4}(0.1) = 2 + 0.025$$

$$= 2.025$$

also

$$\sqrt{4.1} = f(4.1) \approx p_2(4.1)$$

$$= 2 + \frac{1}{4}(4.1-4) - \frac{1}{64}(4.1-4)^2$$

$$= 2 + \frac{1}{4}\cdot\frac{1}{10} - \frac{1}{64}\,\frac{1}{100} = \frac{12959}{6400}$$

$$= 2.02484375$$

Results

For $f(x)=\sqrt{x}$ with $a=4$,

$$p_2(x)=2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2.$$

Using a TI-89 with 12 digits, $\sqrt{4.1}\doteq 2.02484567313$. The difference between this value and our approximations is

approximation	$f(4.1)-p_n(4.1)$
$p_1(4.1)=2.025$	-0.00015432687
$p_2(4.1)=2.02484375$	0.00000192313

The technique used here is common in applied mathematics: When a problem doesn't have a direct method of solution, substitute a *nearby problem* for which a solution can be computed.

A word about notation...

Through out the text, the two symbols

$$\approx \quad \text{and} \quad \doteq$$

will be used to denote approximation. Usually, $\approx$ is used with symbols, e.g.

$$x \approx 0 \quad x \text{ is approximately zero,}$$

and $\doteq$ is used with numbers, e.g.

$$\pi \doteq 3.14159$$

Sometimes it is unclear which symbol is most called for.

Example

The function

$$g(x) = \frac{\ln(1-x)+x}{x^2}, \quad x < 1, \quad x \neq 0$$

is not defined at zero.

(1) Find the approximation $p_4(x)$ to $h(x) = \ln(1-x)$.

(2) Substitute this into g to find a natural way to define $g(0)$.

(3) Compare the result with the limit $\lim_{x\to 0} g(x)$ obtained using l'Hospital's rule.

$h(x) = \ln(1-x)$

$$P_4(x) = h(0) + h'(0)x + \frac{h''(0)}{2!}x^2 + \frac{h'''(0)}{3!}x^3 + \frac{h^{(4)}(0)}{4!}x^4$$

$h(x) = \ln(1-x)$

$h(0) = \ln(1) = 0$

$h'(x) = \frac{-1}{1-x} = -(1-x)^{-1}$

$h'(0) = -1$

$h''(x) = -(1-x)^{-2} = \frac{-1}{(1-x)^2}$

$h''(0) = -1$

$h'''(x) = \frac{-2}{(1-x)^3}$

$h'''(0) = -2$

$h^{(4)}(x) = \frac{-6}{(1-x)^4}$

$h^{(4)}(0) = -6$

$$P_4(x) = -x - \frac{1}{2}x^2 - \frac{2}{3!}x^3 - \frac{6}{4!}x^4$$

$$= -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4$$

For $x \approx 0$ $\quad \ln(1-x) \approx P_4(x)$

So

$$g(x) = \frac{\ln(1-x)+x}{x^2} \approx \frac{P_4(x)+x}{x^2}$$

$$= \frac{-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + x}{x^2}$$

for

$x \approx 0$

$$= \frac{-\frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4}{x^2} = \frac{-1}{2} - \frac{1}{3}x - \frac{1}{4}x^2$$

If $x = 0$, the right side is $\frac{-1}{2}$

So a "natural" way to define $g(0)$ is

$$g(0) = \frac{-1}{2}.$$

Let's compute $\lim_{x \to 0} g(x)$

$$\lim_{x \to 0} \frac{\ln(1-x)+x}{x^2} = \text{"}\frac{0}{0}\text{"}$$ Use l'Hospital's rule

$$= \lim_{x \to 0} \frac{\frac{-1}{1-x} + 1}{2x} = \text{"}\frac{0}{0}\text{"}$$ apply l'Hospital's rule again.

$$= \lim_{x \to 0} \frac{\frac{-1}{(1-x)^2}}{2} = \frac{\frac{-1}{1^2}}{2} = \frac{-1}{2}$$

Section 1.2: Error in Taylor Polynomials

Consider the function $f(x) = \sin x$. The Taylor polynomials of degrees 0 through 6 centered at $a = \frac{\pi}{2}$ are given by

$$\begin{aligned}
p_0(x) &= 1 \\
p_2(x) &= 1 - \frac{\left(x - \frac{\pi}{2}\right)^2}{2} \\
p_4(x) &= 1 - \frac{\left(x - \frac{\pi}{2}\right)^2}{2} + \frac{\left(x - \frac{\pi}{2}\right)^4}{4!} \\
p_6(x) &= 1 - \frac{\left(x - \frac{\pi}{2}\right)^2}{2} + \frac{\left(x - \frac{\pi}{2}\right)^4}{4!} - \frac{\left(x - \frac{\pi}{2}\right)^6}{6!}
\end{aligned}$$

These are plotted together on the next slide.

Plot of f with several Taylor polynomials

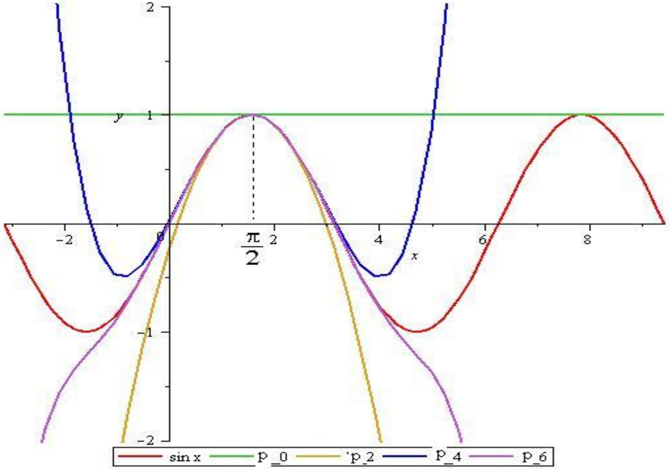

Figure: Plot of f, p_0, p_2, p_4 and p_6 together.

Error

The curves fit very well near the center $\pi/2$, but the approximation breaks down as we move from the center. We can ask the question:

How *good* is our approximation $p_n(x) \approx f(x)$?

It turns out that we can (often) determine the worst case scenario for the error. Not surprisingly, it depends on the degree of the polynomial used as well as the way f behaves.

Taylor's Theorem

Theorem: Suppose f is at least $n+1$ times continuously differentiable on the interval $\alpha \le x \le \beta$, and let a be a point interior to the interval. For the Taylor polynomial p_n centered at a, define the **remainder**, or error in approximating $f(x)$ by $p_n(x)$

$$R_n(x) = f(x) - p_n(x).$$

R_n has 2 parts

Then for each x in $[\alpha, \beta]$

$x - a \to$ how far are we from the center

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c_x),$$

where c_x is some point between a and x.

$f^{(n+1)}(c_x) \to$ how does f behave.

Remark: The number c_x is not known. However, if we can find a bound on $|f^{(n+1)}(t)|$, we know the *worst case* error.

Example

Find an expression for the general Taylor polynomial of degree n for the function $f(x) = e^x$ centered at $a = 0$.And find an expression for the associated error $R_n(x)$

For $a=0$, $p_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n$

$f(x) = e^x \qquad f(0) = e^0 = 1$

$f'(x) = e^x \qquad \vdots$

$f''(x) = e^x$

$\vdots$

$f^{(n)}(x) = e^x \qquad f^{(n)}(0) = e^0 = 1$

$\Rightarrow p_n(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$

$$R_n(x) = \frac{(x-0)^{n+1}}{(n+1)!} f^{(n+1)}(c_x)$$ for some c_x between zero and x

For $f(x) = e^x$, $f^{(n+1)}(x) = e^x$

Hence $$R_n(x) = \frac{x^{n+1}}{(n+1)!} e^{c_x}$$ for some c between zero and x.

As $f(x) = p_n(x) + R_n(x)$, we can write

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} e^{c_x}$$

for some c_x between

zero and x.

Example Continued

Suppose we wish to approximate the number $e = e^1$ using $p_n(x)$ for some degree n. Use Taylor's theorem to find a degree n that will give an accuracy of ten decimal places—i.e. for which

$$R_n(1) \leq 10^{-10}.$$

We know that (for n as yet unknown)

$$R_n(1) = \frac{1^{n+1}}{(n+1)!} e^c \quad \text{for some } c \text{ between } 0 \text{ and } 1$$

For $0 < c < 1$, $e^0 < e^c < e^1$

Since e^x is an increasing function.

Since $e^1 \leq 3$ worst case

$$R_n(1) = \frac{1}{(n+1)!} e^c \leq \frac{1}{(n+1)!} \cdot 3$$

If we choose n such that

$$\frac{3}{(n+1)!} \leq 10^{-10}, \text{ then } R_n(1) \leq 10^{-10}$$

$$\frac{3}{(n+1)!} \leq 10^{-10} \Rightarrow \frac{3}{10^{-10}} \leq (n+1)!$$

$$3 \cdot 10^{10} \leq (n+1)!$$

It turns out that $13! \leq 3 \cdot 10^{10} \leq 14!$.

So we need $n+1 = 14 \Rightarrow n = 13$

we'll satisfy the error bound by taking n at least 13.

Example: $f(x) = p_n(x) + R_n(x)$

Find the Taylor polynomial of degree 2 with the remainder for

$$f(x) = \sqrt[3]{x} \quad \text{centered at } a = 1.$$

$$P_2(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2$$

$$R_2(x) = \frac{(x-1)^3}{3!} f'''(c_x)$$

for some c_x between x and 1.

$f(x) = x^{1/3}$ $\qquad f(1) = 1$

$f'(x) = \frac{1}{3} x^{-2/3}$ $\qquad f'(1) = \frac{1}{3}$

$f''(x) = \frac{-2}{9} x^{-5/3}$ $\qquad f''(1) = \frac{-2}{9}$

$f'''(x) = \frac{10}{27} x^{-8/3}$

$$P_2(x) = 1 + \frac{1}{3}(x-1) - \frac{2/9}{2!}(x-1)^2 = 1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2$$

$$R_2(x) = \frac{(x-1)^3}{3!} \cdot \frac{10}{27} c_x^{-8/3} = \frac{5(x-1)^3}{81} c_x^{-8/3}$$

So

$$\sqrt[3]{x} = 1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{5(x-1)^3 c_x^{-8/3}}{81}$$

for some c_x between 1 and x.