Jan. 20 Math 2254H sec 015H Spring 2015

Recap of Sections 6.2*–6.6

New Differentiation Rules:

$$\frac{d}{dx}\ln x = \frac{1}{x}, \qquad \frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}\ln u = \frac{\frac{du}{dx}}{u}, \qquad \frac{d}{dx}e^u = e^u\,\frac{du}{dx}$$

$$\frac{d}{dx}\log_a u = \frac{\frac{du}{dx}}{(\ln a)u}, \qquad \frac{d}{dx}a^u = (\ln a)a^u\,\frac{du}{dx}$$

$$\frac{d}{dx}\sin^{-1}u = \frac{\frac{du}{dx}}{\sqrt{1-u^2}} = -\frac{d}{dx}\cos^{-1}u$$

$$\frac{d}{dx}\tan^{-1}u = \frac{\frac{du}{dx}}{1+u^2} = -\frac{d}{dx}\cot^{-1}u$$

$$\frac{d}{dx}\sec^{-1}u = \frac{\frac{du}{dx}}{u\sqrt{u^2-1}} = -\frac{d}{dx}\csc^{-1}u$$

New Integration Rules

$$\int \frac{du}{u} = \ln|u| + C, \qquad \int e^u\,du = e^u + C$$

$$\int a^u\,du = \frac{a^u}{\ln a} + C,$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C,$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C,$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C,$$

For the second formula, we assume $a > 0$ and $a \neq 1$. For the last three, we assume $a > 0$.

Examples

Complete the square

$$-x^2+2x+3 = -(x^2-2x+1-1)+3$$
$$= -(x^2-2x+1)+4$$
$$= 4-(x-1)^2$$

(a) $$\int \frac{dx}{\sqrt{3+2x-x^2}}$$

$$= \int \frac{dx}{\sqrt{2^2-(x-1)^2}}$$

$u = x-1$

$du = dx$

$$= \int \frac{du}{\sqrt{2^2-u^2}} = \sin^{-1}\left(\frac{u}{2}\right) + C$$

$$= \sin^{-1}\left(\frac{x-1}{2}\right) + C$$

Examples

(b) $\displaystyle\int_{2\sqrt{2}}^{4} \frac{dx}{x\sqrt{x^2-4}}$ $= \frac{1}{2} \sec^{-1}\left|\frac{x}{2}\right| \Big|_{2\sqrt{2}}^{4}$

$$= \frac{1}{2} \sec^{-1}(2) - \frac{1}{2} \sec^{-1}(\sqrt{2})$$

$$= \frac{1}{2}\left(\frac{\pi}{3}\right) - \frac{1}{2}\left(\frac{\pi}{4}\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{1}{2}\left(\frac{\pi}{12}\right) = \frac{\pi}{24}$$

Some New Limits

The Natural Log:

$$\lim_{x\to 0^+} \ln x = -\infty, \qquad \lim_{x\to\infty} \ln x = \infty$$

Exponentials:

$$\lim_{x\to\infty} a^x = \begin{cases} \infty, & a>1 \\ 0, & 0<a<1 \end{cases}, \qquad \lim_{x\to-\infty} a^x = \begin{cases} 0, & a>1 \\ \infty, & 0<a<1 \end{cases}$$

The Inverse Tangent:

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}, \qquad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

The number e as a Limit:

$$\lim_{x\to 0} (1+x)^{1/x} = e, \qquad \lim_{n\to\infty} \left(1+\frac{r}{n}\right)^n = e^r$$

Section 6.8: Indeterminate Forms & l'Hospital's Rule

We know that both

$$\lim_{x\to 1}(x^2-1)=0 \quad \text{and} \quad \lim_{x\to 1}(x-1)=0.$$

Evaluate $\displaystyle\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}$

$$= \lim_{x\to 1} x+1 = 2$$

And even though $\frac{0}{0}$ is not defined, $\displaystyle\lim_{x\to 1}\frac{x^2-1}{x-1}=2.$

Evaluate Each Limit if Possible

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 3} \frac{x^2 - 9}{(x-3)^2} = \lim_{x \to 3} \frac{(x-3)(x+3)}{(x-3)(x-3)}$$

$$= \lim_{x \to 3} \frac{x+3}{x-3}$$

Does not exist

The *form* $\frac{0}{0}$

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 3} \frac{x^2-9}{(x-3)^2} \quad \text{doesn't exist}$$

We see the *form* $0/0$ when taking this limit. But the limit may or may not exist. And if it does, the form doesn't tell us what the limit might be.

$0/0$ is called an **Indeterminate form**.

Other indeterminate forms we'll encounter include

$$\frac{\pm\infty}{\pm\infty}, \quad \infty-\infty, \quad 0\infty, \quad 1^{\infty}, \quad 0^0, \quad \text{and} \quad \infty^0.$$

Theorem: l'Hospital's Rule

Suppose f and g are differentiable on an open interval I containing a (except possibly at a), and suppose $g'(x) \neq 0$ on I. If

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \lim_{x\to a} g(x) = 0$$

OR if

$$\lim_{x\to a} f(x) = \pm\infty \quad \text{and} \lim_{x\to a} g(x) = \pm\infty$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is ∞ or $-\infty$).

Evaluate each limit if possible

(a) $\lim_{x\to 1} \dfrac{\ln x}{x-1}$ $=$ "$\dfrac{0}{0}$" apply l'H rule

$= \lim_{x\to 1} \dfrac{\frac{1}{x}}{1} = 1$

(b) $\lim_{x\to\infty} xe^{-x}$ $=$ "$\infty \cdot 0$"

$xe^{-x} = \dfrac{e^{-x}}{\frac{1}{x}}$

$= \lim_{x\to\infty} \dfrac{x}{e^{x}} =$ "$\dfrac{\infty}{\infty}$"

or

$xe^{-x} = \dfrac{x}{\frac{1}{e^{-x}}} = \dfrac{x}{e^{x}}$

apply l'H rule

$= \lim_{x\to\infty} \dfrac{1}{e^{x}} = 0$

(c) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ $=$ "$\frac{1-1}{0}$" $=$ "$\frac{0}{0}$"

apply l'H rule

$= \lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x}$

$=$ "$\frac{0}{0}$" use l'H rule again

$= \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$

l'Hospital's Rule is not a "Fix-all"

Evaluate $\lim_{x\to0^+} \frac{\cot x}{\csc x}$ $= "\frac{\infty}{\infty}"$ use l'H rule

$$= \lim_{x\to0^+} \frac{-\csc^2 x}{-\csc x \cot x} = \lim_{x\to0^+} \frac{\csc x}{\cot x} = "\frac{\infty}{\infty}"$$ use l'H rule

$$= \lim_{x\to0^+} \frac{-\csc x \cot x}{-\csc^2 x} = \lim_{x\to0^+} \frac{\cot x}{\csc x}$$

$$\frac{\cot x}{\csc x} = \frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x}} = \cos x \Rightarrow \lim_{x\to0^+} \frac{\cot x}{\csc x} = \lim_{x\to0^+} \cos x = 1$$

Don't apply it if it doesn't apply!

$$\lim_{x \to 2} \frac{x+4}{x^2-3} = \frac{6}{1} = 6$$

BUT

$$\lim_{x \to 2} \frac{\frac{d}{dx}(x+4)}{\frac{d}{dx}(x^2-3)} \quad = \quad \lim_{x \to 2} \frac{1}{2x} \quad = \quad \frac{1}{4}$$

Remarks:

- l'Hopital's rule only applies directly to the forms $0/0$, or $(\pm\infty)/(\pm\infty)$.
- Multiple applications may be needed, or it may not result in a solution.
- It can be applied indirectly to the form $0 \cdot \infty$ by turning the product into a quotient.
- Derivatives of numerator and denominator are taken **separately**–this is NOT a *quotient rule* application.
- Applying it where it doesn't belong likely produces nonsense!

The form $\infty - \infty$

Evaluate the limit if possible

$$\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \text{"}\infty - \infty\text{"} \quad \text{as} \quad x \to 1^+$$

$$= \text{"}-\infty + \infty\text{"} \quad \text{as} \quad x \to 1^-$$

$$= \lim_{x\to 1} \frac{x-1-\ln x}{(x-1)\ln x} = \text{"}\frac{0}{0}\text{"} \quad \text{Use l'H rule}$$

$$= \lim_{x\to 1} \frac{1-\frac{1}{x}}{\ln x + (x-1)\frac{1}{x}} \cdot \frac{x}{x}$$

$$= \lim_{x\to 1} \frac{x-1}{x\ln x + x - 1} = \text{"}\frac{0}{0}\text{"} \quad \text{Use l'H again}$$

$$= \lim_{x \to 1} \frac{1}{\ln x + x \cdot \frac{1}{x} + 1}$$

$$= \lim_{x \to 1} \frac{1}{\ln x + 1 + 1} = \frac{1}{2}$$

Indeterminate Forms 1^∞, 0^0, and ∞^0

Since the logarithm and exponential functions are continuous, and $\ln(x^r) = r\ln x$, we have

$$\lim_{x\to a} F(x) = \exp\left(\ln\left[\lim_{x\to a} F(x)\right]\right) = \exp\left(\lim_{x\to a} \ln F(x)\right)$$

provided this limit exists.

To evaluate $\lim_{x\to a} F(x)$, evaluate

$\lim_{x\to a} \ln F(x)$, then exponentiate.

Use this property to show that

$$\lim_{x\to 0}(1+x)^{1/x} = e$$

$$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = \text{“} 1^{\infty} \text{”}$$

which is equivalent to the form "$1^{-\infty}$"

Take the natural log:

$$\ln (1+x)^{\frac{1}{x}} = \frac{1}{x} \ln(1+x)$$

$$\lim_{x\to 0} \ln(1+x)^{\frac{1}{x}} = \lim_{x\to 0} \frac{\ln(1+x)}{x} = \text{“}\frac{0}{0}\text{”}$$

Use l'H rule

$$\lim_{x\to 0} \ln(1+x)^{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1$$

Exponentiate :

So $$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e^{1} = e$$

Evaluate

$$\lim_{x\to 0^+} x^x = \text{"} 0^0 \text{"}$$

Evaluate $\lim_{x\to 0^+} \ln x^x$

$$\lim_{x\to 0^+} \ln x^x = \lim_{x\to 0^+} x \ln x = \text{"} 0 \cdot (-\infty) \text{"}$$

$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \text{"} \frac{-\infty}{\infty} \text{"}$$

Use l'H rule

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} \frac{1}{x} \cdot (-x^2)$$

$$= \lim_{x \to 0^+} -x = 0$$

Hence $\lim_{x \to 0^+} x^x = e^0 = 1$