January 21 Math 2306 sec 58 Spring 2016

Section 3: First Order Equations: Separation of Variables

The simplest type of equation we could encounter would be of the form

$$\frac{dy}{dx} = g(x).$$

Note

$$y = \int y'\,dx = \int g(x)\,dx$$

For example, solve the ODE

$$\frac{dy}{dx} = 4e^{2x} + 1.$$

$$y = \int y'\,dx = \int (4e^{2x} + 1)\,dx$$

$$= 4 \cdot \frac{e^{2x}}{2} + x + C$$

$$\boxed{y = 2e^{2x} + x + C}$$

one parameter family of solutions

Separable Equations

Definition: The first order equation $y' = f(x, y)$ is said to be **separable** if the right side has the form

$$f(x, y) = g(x)h(y).$$

That is, a separable equation is one that has the form

$$\frac{dy}{dx} = g(x)h(y).$$

Determine which (if any) of the following are separable.

(a) $\dfrac{dy}{dx} = x^3y$

If $g(x) = x^3$ and $h(y) = y$

then the right side is $g(x)\,h(y)$

(b) $\dfrac{dy}{dx} = 2x+y$

There's no way to write $2x+y$ as a product of a function of only x times a function of only y. The DE is <u>not</u> separable.

(c) $\dfrac{dy}{dx} = \sin(xy^2)$ This is not separable.

(d) $\dfrac{dy}{dt} - te^{t-y} = 0$ This is separable

$$\frac{dy}{dt} = te^{t-y} = te^{t} \cdot e^{-y} = g(t)h(y)$$

where $g(t) = te^{t}$ and $h(y) = e^{-y}$

Solving Separable Equations

Recall that from $\frac{dy}{dx} = g(x)$, we can integrate both sides

$$\int \frac{dy}{dx}\,dx = \int g(x)\,dx.$$

This is separable with $h(y) = 1$

we get $y = G(x) + C$

where $G(x)$ is any antiderivative of $g(x)$.

We'll use this observation!

Solving Separable Equations

Let's assume that it's safe to divide by $h(y)$ and let's set $p(y) = 1/h(y)$. We solve (usually find an implicit solution) by **separating the variables**.

$$\frac{dy}{dx} = g(x)h(y) \quad \Longrightarrow \quad \frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

$$\Longrightarrow \quad p(y)\frac{dy}{dx}\,dx = g(x)\,dx \quad \Longrightarrow \quad p(y)\,dy = g(x)\,dx$$

$$\int p(y)\,dy = \int g(x)\,dx \quad \Longrightarrow \quad \underbrace{P(y) = G(x) + C}_{\text{implicit solution}}$$

where P and G are antiderivatives of p and g.

$*$ $\frac{dy}{dx}\,dx = dy$

Solve the ODE

$$\frac{dy}{dx} = -\frac{x}{y} = -x \cdot \frac{1}{y} \quad \Longrightarrow \quad y\,\frac{dy}{dx} = -x$$

$$y\,\frac{dy}{dx}\,dx = -x\;dx \quad \Longrightarrow \quad y\,dy = -x\;dx$$

$$\int y\,dy = \int -x\,dx \quad \Longrightarrow \quad \frac{y^2}{2} = \frac{-x^2}{2} + C$$

This can be arranged as

$$y^2 + x^2 = k$$

where $k = 2C$.

Solve the ODE

$$te^{t-y}\,dt - dy = 0$$

$$dy = t\,e^{t-y}\,dt$$

$$dy = te^{t}\cdot e^{-y}\,dt$$

$$\Rightarrow \frac{1}{e^{-y}}\,dy = te^{t}\,dt$$

$$e^{y}\,dy = te^{t}\,dt$$

$$\int e^{y}\,dy = \int te^{t}\,dt$$

Use parts on the right w/

$u = t \qquad du = dt$

$v = e^{t} \qquad dv = e^{t}\,dt$

$$e^y = te^t - \int e^t\,dt$$

$$\boxed{e^y = te^t - e^t + C}$$

An IVP[1]

$$\frac{dQ}{dt} = -2(Q-70), \quad Q(0) = 180$$

First we solve the ODE.

This is separable.

$$\frac{dQ}{dt} = -2(Q-70) \implies \frac{1}{Q-70}\frac{dQ}{dt} = -2$$

$$\frac{1}{Q-70}\frac{dQ}{dt}\,dt = -2\,dt \implies \frac{1}{Q-70}\,dQ = -2\,dt$$

[1] Recall IVP stands for *initial value problem*.

$$\int \frac{1}{Q-70}\, dQ = \int -2\, dt$$

$$\ln|Q-70| = -2t + C$$

exponentiate $\quad e^{\ln|Q-70|} = e^{-2t+C}$

$$|Q-70| = e^{c} e^{-2t}$$

Let $A = e^{c}$ or $-e^{c}$

$$Q-70 = Ae^{-2t} \implies Q = 70 + Ae^{-2t}$$

Apply the condition $Q(0) = 180$ $\quad 180 = 70 + Ae^{0} = 70 + A$

$\implies A = 110$.

The solution of the IVP is

$$\boxed{Q(t) = 70 + 110\, e^{-2t}}$$

Caveat regarding division by $h(y)$.

Solve the IVP by separation of variables[2]

$$\frac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0$$

Solve the ODE

$$\frac{1}{\sqrt{y}}\frac{dy}{dx} = x \quad \Longrightarrow \quad y^{-1/2}\frac{dy}{dx}\,dx = x\,dx$$

$$\int y^{-1/2}\,dy = \int x\,dx \quad \Longrightarrow \quad \frac{y^{1/2}}{1/2} = \frac{x^2}{2} + C$$

$$y^{1/2} = \frac{x^2}{4} + k \quad \text{where} \quad k = \frac{1}{2}C$$

[2] Remember that one solution is $y(x) = 0$ (for all x).

A family of solutions to the ODE is

$$y = \left(\frac{x^2}{4} + k\right)^2$$

Apply $y(0) = 0$ $\qquad 0 = \left(\frac{0^2}{4} + k\right)^2 = k^2 \implies k = 0$

So "the" solution to the IVP is

$$y = \frac{x^4}{16}$$

The constant solution $y(x) = 0$ was lost when we divided by $\sqrt{y}$ - assuming it was nonzero.

Solutions Defined by Integrals

Recall (Fundamental Theorem of Calculus)

$$\frac{d}{dx}\int_{x_0}^{x} g(t)\,dt = g(x) \quad \text{and} \quad \int_{x_0}^{x} \frac{dy}{dx}\,dx = y(x) - y(x_0).$$

Use this to solve

$$\frac{dy}{dx} = g(x), \quad y(x_0) = y_0$$

$$\int_{x_0}^{x} \frac{dy}{dx}\,dx = \int_{x_0}^{x} g(t)\,dt$$

$$y(x) - y(x_0) = \int_{x_0}^{x} g(t)\,dt$$

$$y(x) - y_0 = \int_{x_0}^{x} g(t)\,dt$$

$$\Rightarrow \boxed{y(x) = y_0 + \int_{x_0}^{x} g(t)\,dt}$$

Note $\frac{d}{dx} y(x) = \frac{d}{dx}\left(y_0 + \int_{x_0}^{x} g(t)\,dt \right)$

$\frac{dy}{dx} = 0 + g(x)$ i.e. $\frac{dy}{dx} = g(x)$

$y(x_0) = y_0 + \int_{x_0}^{x_0} g(t)\,dt = y_0 + 0$ i.e. $y(x_0) = y_0$

Example: Express the solution of the IVP in terms of an integral.

$$\frac{dy}{dx} = \sin(x^2), \quad y(\sqrt{\pi}) = 1$$

Here $g(t) = \sin(t^2)$

$x_0 = \sqrt{\pi}$ and $y_0 = 1$

Using our result

$$y(x) = 1 + \int_{\sqrt{\pi}}^{x} \sin(t^2)\, dt$$