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Section 2: Initial Value Problems

Existence and Uniqueness

Two important questions we can always pose (and sometimes answer) are

(1) Does an IVP have a solution? (existence) and

(2) If it does, is there just one? (uniqueness)

Hopefully it's obvious that we can't solve $\left(\dfrac{dy}{dx}\right)^2+1=-y^2$.

Uniqueness

Consider the IVP

$$\frac{dy}{dx} = x\sqrt{y} \quad y(0) = 0$$

Verify that $y = \frac{x^4}{16}$ is a solution of the IVP. And find a second solution of the IVP by clever guessing.

Does $y = \frac{x^4}{16}$ solve $y(0) = 0$? $\quad y(0) = \frac{0^4}{16} = 0$ yes!

Does it solve $\frac{dy}{dx} = x\sqrt{y}$?

Note $\frac{dy}{dx} = \frac{4x^3}{16} = \frac{x^3}{4}$

and $x\sqrt{y} = x\sqrt{\frac{x^4}{16}} = x\left(\frac{x^2}{4}\right) = \frac{x^3}{4}$

So $\dfrac{dy}{dx} = \dfrac{x^3}{4} = x\sqrt{y}$. It solves the ODE.

Let's find a second solution to

$$\frac{dy}{dx} = x\sqrt{y}, \quad y(0)=0$$

Perhaps a constant function solves it $y=C$.

Since $y(0)=0$, this would require $C=0$.

If $y=0$, then $\dfrac{dy}{dx}=0$ and $x\sqrt{y} = x\sqrt{0} = 0$

Hence $y(x)=0$ is a second solution to the IVP.

Section 3: First Order Equations: Separation of Variables

The simplest type of equation we could encounter would be of the form

$$\frac{dy}{dx} = g(x). \qquad y = \int g(x)\,dx + C$$

For example, solve the ODE

$$\frac{dy}{dx} = 4e^{2x} + 1. \qquad \Rightarrow \quad y = \int \left(4e^{2x}+1\right) dx$$

$$y = 4\cdot\frac{e^{2x}}{2} + x + C$$

$$\boxed{y = 2e^{2x} + x + C}$$

one parameter family of solutions.

Separable Equations

Definition: The first order equation $y' = f(x, y)$ is said to be **separable** if the right side has the form

$$f(x, y) = g(x)h(y).$$

That is, a separable equation is one that has the form

$$\frac{dy}{dx} = g(x)h(y).$$

Determine which (if any) of the following are separable.

(a) $\dfrac{dy}{dx} = x^3 y$

This is separable with

$$g(x) = x^3 \qquad h(y) = y$$

(b) $\dfrac{dy}{dx} = 2x + y$

This is not separable. There is no way to rewrite this as a product of a function of only x and one of only y.

(c) $\frac{dy}{dx} = \sin(xy^2)$ This is not separable.

This is separable.

(d) $\frac{dy}{dt} - te^{t-y} = 0$

$$\frac{dy}{dt} = te^{t-y} = te^{t}e^{-y}$$

here $g(t) = te^{t}$ and $h(y) = e^{-y}$

Solving Separable Equations

Recall that from $\frac{dy}{dx} = g(x)$, we can integrate both sides

$$\int \frac{dy}{dx}\,dx = \int g(x)\,dx.$$

This is separable w/ $h(y) = 1$

one parameter family of solutions

$$y = G(x) + C$$

where $G(x)$ is any antiderivative of $g(x)$.

We'll use this observation!

Solving Separable Equations

Let's assume that it's safe to divide by $h(y)$ and let's set $p(y) = 1/h(y)$. We solve (usually find an implicit solution) by **separating the variables**.

$$\frac{dy}{dx} = g(x)h(y) \quad \Rightarrow \quad \frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

$$* \quad \frac{dy}{dx}\,dx = dy$$

$$p(y)\frac{dy}{dx}\,dx = g(x)\,dx$$

$$\int p(y)\,dy = \int g(x)\,dx \qquad P(y) = G(x) + C$$

one parameter family of solutions given implicitly.

Where $P(y)$ and $G(x)$ are antiderivatives of $p(y)$ and $g(x)$, respectively.

Solve the ODE

$$\frac{dy}{dx} = -\frac{x}{y} = -x \cdot \frac{1}{y} \implies y\,\frac{dy}{dx} = -x \Rightarrow y\frac{dy}{dx}\,dx = -x\,dx$$

$$\int y\,dy = \int -x\,dx \implies \frac{y^2}{2} = \frac{-x^2}{2} + C$$

$$\Rightarrow \quad x^2 + y^2 = k \qquad (k = 2C)$$

Solve the ODE

Let's separate the variables.

$te^{t-y}\,dt - dy = 0$

$$dy = te^{t-y}\,dt = te^{t}e^{-y}\,dt$$

$$\frac{1}{e^{-y}}\,dy = te^{t}\,dt$$

$$\int e^{y}\,dy = \int te^{t}\,dt$$

Use parts on the right

$u = t \qquad du = dt$

$v = e^{t} \qquad dv = e^{t}\,dt$

$$e^{y} = te^{t} - \int e^{t}\,dt$$

$$e^y = te^t - e^t + C$$

An IVP[1]

Solve the ODE, then impose the initial condition.

$$\frac{dQ}{dt} = -2(Q-70), \quad Q(0) = 180$$

$$\frac{1}{Q-70}\frac{dQ}{dt} = -2 \quad \Rightarrow \quad \frac{1}{Q-70}\frac{dQ}{dt}dt = -2\,dt$$

$$\int \frac{1}{Q-70}\,dQ = \int -2\,dt$$

$$\ln|Q-70| = -2t + C$$

Let's find Q explicitly.

[1] Recall IVP stands for *initial value problem*.

$$e^{\ln|Q-70|} = e^{-2t+C} \Rightarrow |Q-70| = e^{C} e^{-2t}$$

Let $A = e^{C}$ or $-e^{C}$

$$Q - 70 = Ae^{-2t} \Rightarrow Q = 70 + Ae^{-2t}$$

Impose the condition $Q(0) = 180$

$$180 = 70 + Ae^{0} = 70 + A \Rightarrow A = 110$$

The solution to the IVP is

$$Q = 70 + 110e^{-2t}.$$

Caveat regarding division by $h(y)$.

Solve the IVP by separation of variables[2]

$$\frac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0$$

$$\frac{1}{\sqrt{y}}\frac{dy}{dx} = x \quad \Longrightarrow \quad y^{-1/2}\frac{dy}{dx}\,dx = x\,dx$$

$$\int y^{-1/2}\,dy = \int x\,dx \qquad \frac{y^{1/2}}{1/2} = \frac{x^2}{2} + C$$

$$y^{1/2} = \frac{x^2}{4} + k \qquad k = \frac{1}{2}C$$

[2] Remember that one solution is $y(x) = 0$ (for all x).

Square to get $y = \left(\frac{x^2}{4} + k\right)^2$

Impose $y(0) = 0$ $\quad 0 = \left(\frac{0^2}{4} + k\right)^2 = k^2 \implies k = 0$

The solution to the IVP is

$$y = \left(\frac{x^2}{4}\right)^2 = \frac{x^4}{16} \quad \text{i.e.} \quad y = \frac{x^4}{16}$$

We lost the solution $y = 0$ when we divided by $\sqrt{y}$ - assuming it was nonzero.

Solutions Defined by Integrals

Recall (Fundamental Theorem of Calculus)

$$\frac{d}{dx}\int_{x_0}^{x} g(t)\,dt = g(x) \quad \text{and} \quad \int_{x_0}^{x} \frac{dy}{dx}\,dx = y(x) - y(x_0).$$

Use this to solve

$$\frac{dy}{dx} = g(x), \quad y(x_0) = y_0$$

$$\frac{dy}{dt} = g(t) \quad \Rightarrow \quad \int_{x_0}^{x} \frac{dy}{dt}\,dt = \int_{x_0}^{x} g(t)\,dt$$

$$y(x) - y(x_0) = \int_{x_0}^{x} g(t)\,dt$$

$$y(x) - y_0 = \int_{x_0}^{x} g(t)\,dt$$

$$\boxed{y(x) = y_0 + \int_{x_0}^{x} g(t)\,dt}$$

Example: Express the solution of the IVP in terms of an integral.

$g(t) = \sin(t^2)$

$$\frac{dy}{dx} = \sin(x^2), \quad y(\sqrt{\pi}) = 1$$

$x_0 = \sqrt{\pi}$, $y_0 = 1$

Using our formulation

$$\boxed{y(x) = 1 + \int_{\sqrt{\pi}}^{x} \sin(t^2)\, dt.}$$