January 21 Math 2335 sec 51 Spring 2016

Section 1.2: Error in Taylor Polynomials

Taylor's Theorem: Suppose f is at least $n+1$ times continuously differentiable on the interval $\alpha \le x \le \beta$, and let a be a point interior to the interval. For the Taylor polynomial p_n centered at a, define the **remainder**, or error in approximating $f(x)$ by $p_n(x)$

$$R_n(x) = f(x) - p_n(x).$$

Then for each x in $[\alpha, \beta]$

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c_x),$$

where c_x is some point between a and x.

Remark: The number c_x is not known. However, if we can find a bound on $|f^{(n+1)}(t)|$, we know the *worst case* error.

Example: $f(x) = p_n(x) + R_n(x)$

Find the Taylor polynomial of degree 2 with the remainder for

$$f(x) = \sqrt[3]{x} \quad \text{centered at } a = 1.$$

Last time, we determined that

$$p_2(x) = 1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2 \quad \text{and} \quad R_2(x) = \frac{5(x-1)^3}{81c_x^{8/3}}$$

where c_x is some number between x and 1.

Bounding Error

When we refer to bounding the error in approximating $f(x)$ by $p_n(x)$, we mean finding some number M such that

$$|f(x) - p_n(x)| \leq M.$$

here x is assumed fixed.

When we refer to bounding the error on an interval $\alpha \leq x \leq \beta$, we mean finding some number K such that

$$|f(x) - p_n(x)| \leq K \quad \text{for all } x \text{ in the interval} \quad \alpha \leq x \leq \beta$$

Example

Use p_2 and R_2 for $f(x) = \sqrt[3]{x}$ found in the previous example to bound the error when p_2 is used to approximate f on the interval $[1/2, 3/2]$.

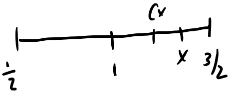
$$R_2(x) = \frac{5(x-1)^3}{81\, c_x^{8/3}} \quad \text{for some } c_x \text{ between } x \text{ and } 1.$$

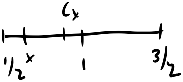
Note $|f(x) - p_2(x)| = |R_2(x)| = \left| \dfrac{5(x-1)^3}{81\, c_x^{8/3}} \right| = \dfrac{5}{81} \dfrac{|x-1|^3}{c_x^{8/3}}$

For $\frac{1}{2} \leq x \leq \frac{3}{2} \implies \frac{1}{2} - 1 \leq x-1 \leq \frac{3}{2} - 1 \implies \frac{-1}{2} \leq x-1 \leq \frac{1}{2}$

i.e. $|x-1| \leq \frac{1}{2}$

so the largest $|x-1|^3$ can be is $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$





Since c_x must be between 1 and x we have

$$\frac{1}{2} \le c_x \le \frac{3}{2}$$

For $\frac{1}{2} \le c_x \le \frac{3}{2}$, $\left(\frac{1}{2}\right)^{8/3} \le c_x^{8/3} \le \left(\frac{3}{2}\right)^{8/3}$

Taking reciprocals $\left(\frac{3}{2}\right)^{-8/3} \le c_x^{-8/3} \le \left(\frac{1}{2}\right)^{-8/3}$

So $c_x^{-8/3} \le \left(\frac{1}{2}\right)^{-8/3} = 2^{8/3} = 4 \cdot 2^{2/3}$

That is, $\dfrac{1}{c_x^{8/3}}$ is at most $4 \cdot 2^{2/3}$

Hence $|R_2(x)| = \frac{5}{81}\frac{|x-1|^3}{c_x^{8/3}} \le \frac{5}{81}\left(\frac{1}{8}\right)\cdot 4\cdot 2^{2/3}$

$$\doteq 0.048994$$

If we use $P_2(x)$ to approximate $\sqrt[3]{x}$ for any x between $\frac{1}{2}$ and $\frac{3}{2}$ the error will be at most 0.048994.

Well Known Taylor Polynomials with Remainders

See page 13 equations (1.13)–(1.17).

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}e^c$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + (-1)^n\frac{x^{2n+1}}{(2n+1)!}\cos(c)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + (-1)^{n+1}\frac{x^{2n+2}}{(2n+2)!}\cos(c)$$

Continued...

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \frac{x^{n+1}}{1-x} \quad x \neq 1 \quad \text{(an exact formula)}$$

$$(1+x)^\alpha = 1 + \binom{\alpha}{1} x + \binom{\alpha}{2} x^2 + \cdots + \binom{\alpha}{2} x^n + \binom{\alpha}{n+1} x^{n+1} (1+c)^{\alpha-(n+1)}$$

Here, α is a real number,

$$\binom{\alpha}{0} = 1, \quad \text{and} \quad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}$$

Note $\alpha - k + 1 = \alpha - (k-1)$ e.g $\binom{\alpha}{1} = \frac{\alpha}{1!}$ $\binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2!}$

$\binom{\alpha}{3} = \frac{\alpha(\alpha-1)(\alpha-2)}{3!}$ and so on.

Example

Use the last formulation to express $p_3(x) + R_3(x)$ for

$$f(x) = (1+x)^{3/2}.$$

Here, $\alpha = \frac{3}{2}$

$$P_3(x) + R_3(x) = \underbrace{1 + \binom{3/2}{1}x + \binom{3/2}{2}x^2 + \binom{3/2}{3}x^3}_{P_3} + \underbrace{\binom{3/2}{4}x^4(1+c)^{\frac{3}{2}-4}}_{R_3}$$

$$\binom{3/2}{1} = \frac{3/2}{1!} = \frac{3}{2}, \quad \binom{3/2}{2} = \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2!} = \frac{\frac{3}{2}\cdot\frac{1}{2}}{2} = \frac{3}{8}$$

$$\binom{3/2}{3} = \frac{3/2\,(3/2-1)(3/2-2)}{3!} = \frac{3/2\,(3/2-1)(3/2-2)}{2!\cdot 3} = \frac{3}{8}\ \frac{-1/2}{3} = \frac{-1}{16}$$

$$\binom{3/2}{4} = \frac{3/2\,(3/2-1)(3/2-2)(3/2-3)}{4!} = \frac{3/2\,(3/2-1)(3/2-2)}{3!}\cdot\frac{(3/2-3)}{4} = \frac{-1}{16}\cdot\frac{-3/2}{4} = \frac{3}{128}$$

So

$$P_3(x) = 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3$$

and

$$R_3(x) = \frac{3}{128}x^4\,(1+c)^{-5/2}.$$

for some c between x and 0.

Bounding Error

$* \quad R_{2n}(x) = (-1)^{n+1} \frac{x^{2n+2}}{(2n+2)!} \cos C$

Use the remainder term for $f(x) = \cos x$ to bound the error when $p_2(x) = 1 - x^2/2$ is used to approximate f on the interval $[-\pi/4, \pi/4]$.

If $2n = 2$, then $n = 1$

$$R_2(x) = (-1)^{1+1} \frac{x^{2+2}}{(2+2)!} \cos C = \frac{x^4}{4!} \cos C$$

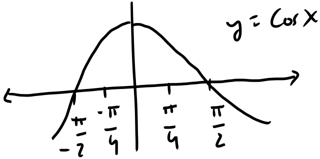
for some c between 0 and x.

$$|R_2(x)| = \frac{|x|^4}{4!} |\cos c|$$

For $-\pi/4 \le x \le \pi/4$

$|x| \le \pi/4$

So $|x|^4 \le \left(\frac{\pi}{4}\right)^4$



For $\frac{-\pi}{4} \le c \le \frac{\pi}{4}$

$$\frac{1}{\sqrt{2}} \le |\cos c| \le 1$$

Hence $|R_2(x)| = \frac{1}{24} |x|^4 |\cos c| \le \frac{1}{24}\left(\frac{\pi}{4}\right)^4 \cdot 1$

$$\doteq 0.015854$$

According to TI-89

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \doteq 0.707107$$

$$1 - \frac{1}{2}\left(\frac{\pi}{4}\right)^2 \doteq 0.691575$$

$$\cos\frac{\pi}{4} - P_2\left(\frac{\pi}{4}\right) = 0.015532$$

New Polynomials from Old

Recall: $$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \frac{x^{n+1}}{1-x} \quad x \neq 1$$

(a) Use the substitution $x = -t^2$ to find a polynomial and remainder for

$$f(t) = \frac{1}{1+t^2}.$$

$$\frac{1}{1+t^2} = \frac{1}{1-(-t^2)} = 1 + (-t^2) + (-t^2)^2 + (-t^2)^3 + \dots + (-t^2)^n + \frac{(-t^2)^{n+1}}{1-(-t^2)}$$

$$= 1 - t^2 + t^4 - t^6 + \dots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}$$

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \ldots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}$$

for $-t^2 \neq 1$

(note $-t^2 \neq 1$ for all real t.)

A Mean Value Theorem

Theorem: (Integral Mean Value Theorem) Let $w(x)$ be a nonnegative integrable function on (a, b) and let $f(x)$ be continuous on $[a, b]$. Then there exists a point c in $[a, b]$ such that

$$\int_a^b f(x)w(x)\,dx = f(c)\int_a^b w(x)\,dx$$

Special case: If $w(x)=1$, then $\int_a^b w(x)\,dx = \int_a^b dx = b-a$

In this case, it reduces to the familiar

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

Example of the Mean Value Theorem

For integer $k \geq 0$

$$\int_0^x \frac{t^k}{1+t^2}\,dt = \frac{1}{1+c^2}\int_0^x t^k\,dt$$

for some c between 0 and x.

Here, $f(t) = \dfrac{1}{1+t^2}$ and $w(t) = t^k$

Note $\displaystyle\int_0^x \frac{t^k}{1+t^2}\,dt = \frac{1}{1+c^2}\left[\frac{t^{k+1}}{k+1}\right|_0^x = \frac{1}{1+c^2}\cdot\frac{x^{k+1}}{k+1}$

for some c between 0 and x.

Example

(b) Use the results for the function $f(t) = \frac{1}{1+t^2}$, and the fact that

$$\tan^{-1}(x) = \int_0^x \frac{dt}{1+t^2}$$

to find a Taylor polynomial with remainder for the function $g(x) = \tan^{-1}(x)$.

We had $\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \dots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}$

so $\int_0^x \frac{1}{1+t^2}\,dt = \int_0^x \left(1 - t^2 + t^4 - t^6 + \dots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\right) dt$

$$\tan^{-1} x = \int_0^x \left(1 - t^2 + t^4 - t^6 + \dots + (-1)^n t^{2n}\right) dt + \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\,dt$$

$$\tan^{-1}x = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \dots + (-1)^n \frac{t^{2n+1}}{2n+1} \Bigg|_0^x + \frac{(-1)^{n+1}}{1+c^2} \int_0^x t^{2n+2}\,dt$$

MVT

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \frac{(-1)^{n+1}}{1+c^2} \cdot \frac{x^{2n+3}}{2n+3}$$

for some c between 0 and x.

So for $\tan^{-1}x$

$$P_{2n+1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1}$$

and

$$R_{2n+1}(x) = \frac{(-1)^{n+1}}{1+c^2} \frac{x^{2n+3}}{2n+3}$$ for c between 0 and x.

Using Taylor Polynomials

Use a Taylor polynomial with remainder to evaluate the limit

$$\lim_{x\to 0} \frac{1+x^2-e^{x^2}}{x^4}$$

From before

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots + \frac{t^n}{n!} + \frac{t^{n+1}}{(n+1)!}e^c$$

for some c between 0 and t

Taking $t = x^2$

$$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \dots + \frac{(x^2)^n}{n!} + \frac{(x^2)^{n+1}}{(n+1)!}e^c$$

for some c where $0 < c < x^2$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots + \frac{x^{2n}}{n!} + \frac{x^{2n+2}}{(n+1)!} e^c$$

$$\frac{1 + x^2 - e^{x^2}}{x^4} = \frac{1 + x^2 - \left(1 + x^2 + \frac{x^4}{2!} + \dots + \frac{x^{2n}}{n!} + \frac{x^{2n+2}}{(n+1)!} e^c\right)}{x^4}$$

$$= \frac{-\frac{x^4}{2!} - \frac{x^6}{3!} - \dots - \frac{x^{2n}}{n!} - \frac{x^{2n+2}}{(n+1)!} e^c}{x^4}$$

$$= -\frac{1}{2} - \frac{x^2}{3!} - \dots - \frac{x^{2n-4}}{n!} - \frac{x^{2n-2}}{(n+1)!} e^c$$

Take $x \to 0$

$$\lim_{x\to 0} \frac{1+x^2-e^{x^2}}{x^4} = \lim_{x\to 0}\left(\frac{-1}{2} - \frac{x^2}{3!} - \cdots - \frac{x^{2n-4}}{n!} - \frac{x^{2n-2}}{(n+1)!}e^c\right)$$

$$= \frac{-1}{2}$$