Jan. 22 Math 2254H sec 015H Spring 2015

Section 7.1: Integration by Parts

Recall the product rule:

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

Let's subtract gf' from both sides and consider an anti-derivative:

$$\begin{aligned}
f(x)g'(x) &= \frac{d}{dx}[f(x)g(x)] - g(x)f'(x) \quad \Longrightarrow \\
\int f(x)g'(x)\,dx &= \int \left\{ \frac{d}{dx}[f(x)g(x)] - g(x)f'(x) \right\} dx \\
&= \int \frac{d}{dx}[f(x)g(x)]\,dx - \int g(x)f'(x)\,dx
\end{aligned}$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

let $u = f(x)$, then $du = f'(x)\,dx$

$v = g(x)$, then $dv = g'(x)\,dx$

$$\int u\,dv = uv - \int v\,du$$

$\int u\,dv = uv - \int v\,du$

Evaluate

$$\int x \cos x \, dx$$

Choices:

$u = x$ and $dv = \cos x \, dx$

or

$u = \cos x$ and $dv = x \, dx$

Let $u = x$ $\quad du = dx$

$v = \sin x$ $\quad dv = \cos x \, dx$

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx$$

$$= x \sin x + \cos x + C$$

$\int u\,dv = uv - \int v\,du$

Evaluate

$$\int y^2 e^y\,dy$$

Let $u = y^2$ $\quad du = 2y\,dy$

$v = e^y$ $\quad dv = e^y\,dy$

$u = 2y$ $\quad du = 2dy$

$v = e^y$ $\quad dv = e^y\,dy$

$$= y^2e^y - \int 2y\,e^y\,dy$$

$$= y^2e^y - \left(2ye^y - \int 2e^y\,dy\right)$$

$$= y^2e^y - 2ye^y + 2e^y + C$$

$\int u\,dv = uv - \int v\,du$

Evaluate

$$\int x^3 \ln x \, dx$$

Let $u = x^3 \qquad du = 3x^2\,dx$

$v = ? \qquad dv = \ln x \, dx$

Try again

$u = \ln x \qquad du = \frac{1}{x}\,dx$

$v = \frac{x^4}{4} \qquad dv = x^3\,dx$

$$= \frac{x^4}{4}\ln x - \int \frac{x^4}{4}\cdot\frac{1}{x}\,dx$$

$$= \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3\,dx$$

$$= \frac{x^4}{4}\ln x - \frac{1}{16}x^4 + C$$

Another unusual use of Int. by Parts

Evaluate the integral $\mathcal{I}$ where

$$\mathcal{I} = \int e^x \cos x \, dx$$

Let $u = e^x$ $\quad du = e^x \, dx$

$v = \sin x$ $\quad dv = \cos x \, dx$

$$= e^x \sin x - \int e^x \sin x \, dx \; *$$

Let $u = e^x$ $\quad du = e^x \, dx$

$v = -\cos x$ $\quad dv = \sin x \, dx$

$$= e^x \sin x - \left(-e^x \cos x + \int e^x \cos x \, dx \right)$$

$$\mathcal{I} = e^x \sin x + e^x \cos x - \mathcal{I} + 2C$$

Add I to both sides to get

$$2I = e^x \sin x + e^x \cos x + 2C$$

$$\Rightarrow \boxed{I = \frac{1}{2} e^x \sin x + \frac{1}{2} e^x \cos x + C}$$

* $$\int e^x \sin x \, dx = e^x \sin x - I$$

$$= \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x + K$$

Definite integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)|_a^b - \int_a^b g(x)f'(x)\,dx$$

Evaluate $\int_1^e \ln x\,dx$ $=$

$u = \ln x \quad du = \frac{1}{x}\,dx$

$v = x \quad dv = dx$

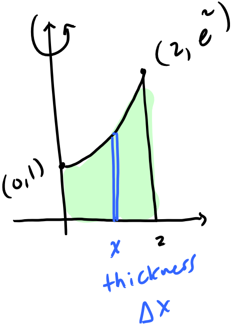
$$x \ln x \Big|_1^e - \int_1^e x \cdot \frac{1}{x}\,dx$$

$$= x \ln x \Big|_1^e - \Big[x \Big|_1^e$$

$$= e \ln e - 1 \ln 1 - [e - 1] = e - e + 1 = 1$$

Application

The region in the plane bounded between the curves $y = e^x$, $x = 2$, $y = 0$ and $x = 0$ is rotated about the y-axis to form a solid. Find the volume of the solid.



radius

$r = x$

height

$h = e^x$

Volume of one shell

$$V_s = 2\pi r h \Delta x$$

$$= 2\pi x e^x \Delta x$$

Total Volume

$$V = \int_0^2 2\pi x e^x \, dx$$

$u = x \qquad du = dx$

$v = e^x \qquad dv = e^x \, dx$

$$= 2\pi \left[x e^x \Big|_0^2 - \int_0^2 e^x \, dx \right]$$

$$= 2\pi \left[x e^x \Big|_0^2 - e^x \Big|_0^2 \right]$$

$$= 2\pi \left[2e^2 - 0 - (e^2 - e^0) \right]$$

$$= 2\pi (e^2 + 1)$$