January 22 Math 3260 sec. 55 Spring 2020

Section 1.3: Vector Equations

We defined vectors in $\mathbb{R}^n$ along with two operations. Scalars in our context are **real numbers.**

For $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and scalar c

- Scalar multiplication $c\mathbf{x} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}$.
- Vector Addition: $\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$

Algebraic Properties on $\mathbb{R}^n$

The **zero vector** in $\mathbb{R}^n$ is the n-tuple of all zeros. It is denoted $\mathbf{0}$ (or $\vec{0}$).

For every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^n$ and scalars c and d[1]

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(iv) $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$

(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(vii) $c(d\mathbf{u})=d(c\mathbf{u})=(cd)\mathbf{u}$

(viii) $1\mathbf{u}=\mathbf{u}$

[1] The term $-\mathbf{u}$ denotes $(-1)\mathbf{u}$.

Definition: Linear Combination

A linear combination of vectors $\mathbf{v}_1, \dots \mathbf{v}_p$ in $\mathbb{R}^n$ is a vector $\mathbf{y}$ of the form

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

where the scalars $c_1, \dots, c_p$ are often called weights.

For example, suppose we have two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Some linear combinations include

$$3\mathbf{v}_1, \quad -2\mathbf{v}_1 + 4\mathbf{v}_2, \quad \frac{1}{3}\mathbf{v}_2 + \sqrt{2}\mathbf{v}_1, \quad \text{and} \quad \mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2.$$

zero vector $\vec{0}$

scalar zero

Example

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$. Determine if $\mathbf{b}$ can be written as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

Q: Are there numbers (scalars) x_1 and x_2 such that $x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$?

Set up the equation:

$$x_1 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \overset{?}{=} \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ -2x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 0 \\ 2x_2 \end{bmatrix} \overset{?}{=} \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$$

use algebra defined

$$\begin{bmatrix} x_1 + 3x_2 \\ -2x_1 \\ -x_1 + 2x_2 \end{bmatrix} \overset{?}{=} \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$$

requires

$$\begin{aligned} x_1 + 3x_2 &= -2 \\ -2x_1 \quad\quad &= -2 \\ -x_1 + 2x_2 &= -3 \end{aligned}$$

a linear system of equations

Let's determine if the system is consistent by using an augmented matrix.

$$\begin{bmatrix} 1 & 3 & -2 \\ -2 & 0 & -2 \\ -1 & 2 & -3 \end{bmatrix} \xrightarrow{\text{rref (by TI)}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

not a pivot column

The system is consistent, so $\vec{b}$ is a linear combination of $\vec{a}_1$ and $\vec{a}_2$

In fact, we see that $x_1 = 1$ and $x_2 = -1$

so $\vec{b} = \vec{a}_1 - \vec{a}_2$.

Some Convenient Notation

Letting $\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}$, and in general $\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$, for $j = 1, ..., n$, we can denote the $m \times n$ matrix whose columns are these vectors by

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Note that each vector $\mathbf{a}_j$ is a vector in $\mathbb{R}^m$.

Vector and Matrix Equations

The vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}. \tag{1}$$

In particular, **b** is a linear combination of the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if the linear system whose augmented matrix is given in (1) is consistent.

Definition of **Span**

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set of vectors in $\mathbb{R}^n$. The set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = \text{Span}(S).$$

It is called the **subset of $\mathbb{R}^n$ spanned by (a.k.a. generated by)** the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

To say that a vector $\mathbf{b}$ is in $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ means that there exists a set of scalars $c_1, \ldots, c_p$ such that $\mathbf{b}$ can be written as

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p.$$

Span

If $\mathbf{b}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, then $\mathbf{b} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$. From the previous result, we know this is equivalent to saying that the vector equation

$$x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$$

has a solution. This is in turn the same thing as saying the linear system with augmented matrix $[\mathbf{v}_1 \ \cdots \ \mathbf{v}_p \ \mathbf{b}]$ is consistent.

Examples

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, and $\mathbf{a}_2 = \begin{bmatrix} -1 \\ 4 \\ -2 \end{bmatrix}$.

(a) Determine if $\mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ is in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$.

Q: Is $\vec{b}$ a linear combination of $\vec{a}_1$ and $\vec{a}_2$?

Is $x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$ a consistent vector equation?

Is the system whose augmented matrix is

$[\vec{a}_1 \ \vec{a}_2 \ \vec{b}]$ consistent?

Let's use an rref: $[\vec{a}_1 \ \vec{a}_2 \ \vec{b}] = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & 2 \\ 2 & -2 & 1 \end{bmatrix}$

$$\xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

pivot column

The system is inconsistent, hence $\vec{b}$ is <u>not</u> in $\text{Span}\{\vec{a}_1, \vec{a}_2\}$.

(b) Determine if $\mathbf{b} = \begin{bmatrix} 5 \\ -5 \\ 10 \end{bmatrix}$ is in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$.

$$[\vec{a}_1 \ \vec{a}_2 \ \vec{b}] = \begin{bmatrix} 1 & -1 & 5 \\ 1 & 4 & -5 \\ 2 & -2 & 10 \end{bmatrix}$$

$$\xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

$\therefore$ not a pivot column $\therefore$ system is consistent!

$\vec{b}$ is in span $\{\vec{a}_1, \vec{a}_2\}$. Moreover

$\vec{b} = 3\vec{a}_1 - 2\vec{a}_2$

Another Example

Give a geometric description of the subset of $\mathbb{R}^2$ given by
Span$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

$\vec{u}$ is in Span$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ if $\vec{u} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$

for some number x_1.

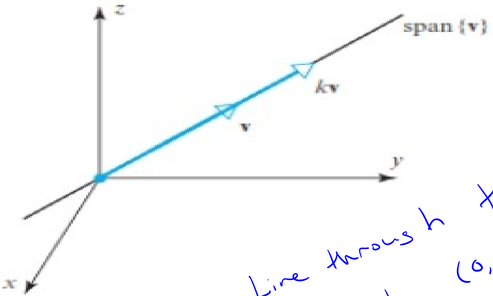
So this is all points of the form

$(x_1, 0)$ for all possible x_1.

This is the x-axis in $\mathbb{R}^2$.

Span$\{\mathbf{v}\}$ in $\mathbb{R}^3$

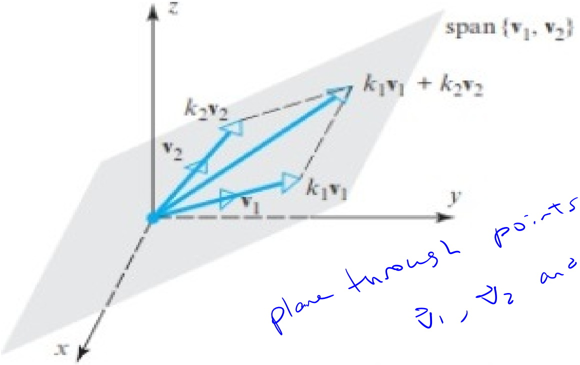
If $\mathbf{v}$ is any nonzero vector in $\mathbb{R}^3$, then Span$\{\mathbf{v}\}$ is a line through the origin parallel to $\mathbf{v}$.



Line through the point $\vec{v}$ and $(0,0,0)$.

Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ in $\mathbb{R}^3$

If $\mathbf{v}_1$ and $\mathbf{v}_2$ are nonzero, and nonparallel vectors in $\mathbb{R}^3$, then Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane containing the origin parallel to both vectors.



plane through points $\vec{v}_1, \vec{v}_2$ and $(0,0,0)$.

Example

Let $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (0, 2)$ in $\mathbb{R}^2$. Show that for every pair of real numbers a and b, that (a, b) is in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$.

we need to show that $x_1\vec{u} + x_2\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$

is always consistent — i.e for all (a, b).

$$\begin{bmatrix} \vec{u} & \vec{v} & \begin{bmatrix} a \\ b \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 & a \\ 1 & 2 & b \end{bmatrix}$$

$$-R_1 + R_2 \to R_2 \qquad \begin{bmatrix} 1 & 0 & a \\ 0 & 2 & b - a \end{bmatrix}$$

$\frac{1}{2}R_2 \to R_2$

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & \frac{b-a}{2} \end{bmatrix}$$

always consistent
never a pivot column

In fact

$$\begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{b-a}{2} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$