Jan. 23 Math 2254H sec 015H Spring 2015

Section 7.2: Trigonometric Integrals

Compare the two integrals

$$\int \cos^3 x\, dx$$

and

Let $u = \sin x$, $du = \cos x\, dx$

$$\int (1-\sin^2 x) \cos x\, dx = \int (1-u^2)\, du = u - \frac{u^3}{3} + C$$

$$= \sin x - \frac{\sin^3 x}{3} + C$$

Evaluate $\int \sin^2 x \cos^3 x \, dx$ $= \int \sin^2 x \,(1-\sin^2 x) \cos x \, dx$

$u = \sin x, \quad du = \cos x \, dx$

$$\int u^2(1-u^2)\, du = \int (u^2 - u^4)\, du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

$\int \sin^m x \cos^n x \, dx$

(a) If n is odd ($n = 2k + 1$), then save one cosine for du, write the remaining cosines as

$$\cos^{2k} x = (1 - \sin^2 x)^k,$$

and choose the substitution $u = \sin x$.

$$\int \sin^m x \cos^n x \, dx = \int \sin^m x (1 - \sin^2 x)^k \cos x \, dx$$

$$= \int u^m (1 - u^2)^k \, du$$

$\int \sin^m x \cos^n x\, dx$

(b) If m is odd ($m = 2p + 1$), then save one sine for du, write the remaining sines as

$$\sin^{2p} x = (1 - \cos^2 x)^p,$$

and choose the substitution $u = \cos x$.

$$\int \sin^m x \cos^n x\, dx = \int \sin x (1 - \cos^2 x)^p \cos^n x\, dx$$

$$= -\int u^n (1 - u^2)^p\, du$$

Evaluate

$$\int \sin^5 x\,dx = -\cos x + \frac{2}{3}\cos^3 x - \frac{\cos^5 x}{5} + C$$

Evaluate

$$\int \sin^3 x \cos^3 x \, dx = \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C$$

$$= \frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + K$$

What if both m and n are even?

Recall the half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad \text{and} \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x).$$

$\int \sin^m x \cos^n x \, dx$

(c) If both m and n are even, use the half-angle identities above. The identity

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

may also be useful.

Evaluate

$$\int \cos^4\theta\, d\theta$$

$$\cos^4\theta = \left(\cos^2\theta\right)^2$$

$$= \left(\tfrac{1}{2}(1+\cos 2\theta)\right)^2$$

$$= \tfrac{1}{4}\left(1 + 2\cos 2\theta + \cos^2 2\theta\right)$$

$$= \tfrac{1}{4}\left(1 + 2\cos 2\theta + \tfrac{1}{2} + \tfrac{1}{2}\cos 4\theta\right)$$

$$= \tfrac{1}{4}\left(\tfrac{3}{2} + 2\cos 2\theta + \tfrac{1}{2}\cos 4\theta\right)$$

$$\int \cos^4\theta\, d\theta = \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \frac{1}{4}\left(\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8}\sin 4\theta\right) + C$$

Other Uses of Trig Identities

Recall:

$$\frac{d}{dx}\tan x = \sec^2 x \quad \text{and} \quad \frac{d}{dx}\sec x = \sec x \tan x$$

$$tan^2 x + 1 = \sec^2 x$$

$$\int \tan x\, dx = \ln|\sec x| + C$$

and

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C$$

Evaluate

$$\int \sec^4\theta \tan\theta \, d\theta = \frac{\tan^4\theta}{4} + \frac{\tan^2\theta}{2} + C$$

$$= \frac{\sec^4\theta}{4} + K$$

Evaluate

$$\int \tan^3 t\, dt = \int (\sec^2 t - 1) \tan t \; dt$$

$$= \int \sec^2 t \tan t \; dt - \int \tan t \; dt$$

$$= \int u\, du - \ln|\sec t| + C$$

$u = \tan t$

$du = \sec^2 t\, dt$

$$= \frac{u^2}{2} - \ln|\sec t| + C$$

$$= \frac{\tan^2 t}{2} - \ln|\sec t| + C$$