### January 23 Math 2306 sec. 53 Spring 2019

#### Section 4: First Order Equations: Linear

We wish to solve a first order linear equation. Such an equation in standard form looks like

$$\frac{dy}{dx} + P(x)y = f(x).$$

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We obtain a **general solution** in the form  $y = y_c + y_p$  using an integrating factor.

### General Solution of First Order Linear ODE

- ► Put the equation in standard form y' + P(x)y = f(x), and correctly identify the function P(x).
- Obtain the integrating factor  $\mu(x) = \exp(\int P(x) dx)$ .
- Multiply both sides of the equation (in standard form) by the integrating factor µ. The left hand side will always collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)} \int \mu(x) f(x) \, dx = e^{-\int P(x) \, dx} \left( \int e^{\int P(x) \, dx} f(x) \, dx + C \right)$$

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# Solve the ODF The ODE is in standard form, $\frac{dy}{dx} + y = 3xe^{-x}$ P(x) = 1Build the integrating factor $\mu = e$ μ= e = p, We can take the constant of integration to be anything. I'm taking it to be zero. Multiply the ODE by M

 $e^{*} \frac{d_{y}}{dx} + e^{*} y = e^{*} (3 \times e^{*})$ 

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$$\frac{d}{dx} \begin{pmatrix} x \\ ey \end{pmatrix} = 3x e^{x} e^{x} = 3x$$
Integrate
$$\int \frac{d}{dx} (e^{x}y) dx = \int 3x dx$$

$$e^{x}y = \frac{3}{2}x^{2} + C$$
The constant of integration must appear at this step.
The general solution is
$$y = \frac{3}{2}x^{2} + C$$

$$y = \frac{3}{2}x^{2} e^{x} + C e^{x}$$

$$\int c = Ce^{x}$$

$$\int p = \frac{3}{2}x^{2}e^{x}$$

$$\int c = 2e^{x}$$

# Solve the IVP

$$\vec{x}' \frac{dy}{dx} - \vec{x}' \left(\frac{1}{x}\right) y = \vec{x}' (2x)$$

$$\frac{d}{dx} (\vec{x}' y) = 2$$

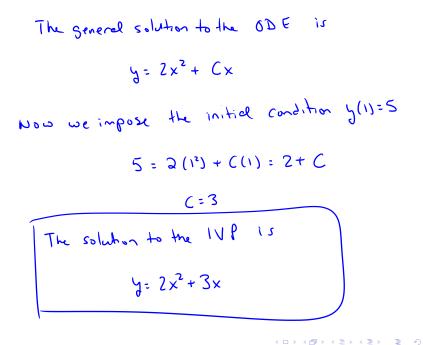
$$\int \frac{d}{dx} (\vec{x}' y) dx = \int 2 dx$$

$$\vec{x}' y = 2x + C$$

$$y = \frac{2x + C}{x^{-1}}$$

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# Verify

Just for giggles, lets verify that our solution  $y = 2x^2 + 3x$  really does solve the differential equation we started with

$$x\frac{dy}{dx} - y = 2x^{2}.$$
  
Let's substitute into the ODE.  $y' = 4x + 3$   
$$x(4x+3) - (2x^{2} + 3x) =$$
  
$$4x^{2} + 3x - 2x^{2} - 3x =$$
  
$$4x^{2} - 2x^{2} =$$
  
$$2x^{2} = 2x^{2}$$
  
Ta Da

# Steady and Transient States

For some linear equations, the term  $y_c$  decays as x (or t) grows. For example

$$\frac{dy}{dx} + y = 3xe^{-x}$$
 has solution  $y = \frac{3}{2}x^2e^{-x} + Ce^{-x}$ .  
Here,  $y_p = \frac{3}{2}x^2e^{-x}$  and  $y_c = Ce^{-x}$ .

Such a decaying complementary solution is called a transient state.

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The corresponding particular solution is called a steady state.

# **Bernoulli Equations**

Suppose P(x) and f(x) are continuous on some interval (a, b) and n is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is called a Bernoulli equation.

**Observation:** This equation has the flavor of a linear ODE, but since  $n \neq 0, 1$  it is necessarily nonlinear. So our previous approach involving an integrating factor does not apply directly. Fortunately, we can use a change of variables to obtain a related linear equation.