January 23 Math 2306 sec. 54 Spring 2019

Section 4: First Order Equations: Linear

We wish to solve a first order linear equation. Such an equation in standard form looks like

$$\frac{dy}{dx} + P(x)y = f(x).$$

We obtain a **general solution** in the form $y = y_c + y_p$ using an integrating factor.

General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx}f(x)\,dx + C\right)$$

Solve the ODE

$$\frac{dy}{dx}+y=3xe^{-x}$$

The ODE is in standard form.

$P(x)=1$

Build the integrating factor $\mu = e^{\int P(x)\,dx}$

$\mu = e^{\int dx} = e^{x}$

We can take the constant of integration to be any number.
I'll take it to be zero.

Multiply the ODE by μ

$$e^{x}\frac{dy}{dx}+e^{x}y = e^{x}\left(3xe^{-x}\right) = 3xe^{-x}e^{x} = 3x$$

$$\frac{d}{dx}\left(e^{x} y\right) = 3x$$

$$\int \frac{d}{dx}\left(e^{x} y\right) dx = \int 3x \, dx$$

$$e^{x} y = \frac{3}{2}x^2 + C$$

The constant of integration must appear at this step.

$$y = \frac{\frac{3}{2}x^2 + C}{e^{x}}$$

$$y_c = Ce^{-x}$$

$$y = \frac{3}{2}x^2 e^{-x} + Ce^{-x}$$

$$y_p = \frac{3}{2}x^2 e^{-x}$$

Solve the IVP

$$x\frac{dy}{dx} - y = 2x^2,\ x > 0 \quad y(1) = 5$$

The ODE is not in standard form.

Divide by x:

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{2x^2}{x} = 2x \qquad P(x) = \frac{-1}{x}$$

Integrating factor $\mu = e^{\int P(x)dx} = e^{\int \frac{-1}{x}dx} = e^{-\ln x}$

$$\mu = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}$$

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x}\left(\frac{1}{x}\right)y = \frac{1}{x}(2x)$$

$$\frac{d}{dx}\left(\frac{1}{x}y\right) = 2$$

$$\int \frac{d}{dx}\left(\frac{1}{x}y\right)dx = \int 2\,dx$$

$$\frac{1}{x}y = 2x + C$$

$$y = \frac{2x+C}{1/x} = 2x^2 + Cx$$

The general solution to the ODE is

$$y = 2x^2 + Cx$$

Now apply the initial condition $y(1) = 5$.

$$5 = 2(1^2) + C(1) = 2 + C$$

$$C = 3$$

The solution to the IVP is

$$y = 2x^2 + 3x$$

Verify

Just for giggles, lets verify that our solution $y = 2x^2 + 3x$ really does solve the differential equation we started with

$$x\frac{dy}{dx} - y = 2x^2.$$

Let's substitute into the left side. $\frac{dy}{dx} = 4x + 3$

$$x(4x+3) - (2x^2 + 3x) =$$

$$4x^2 + 3x - 2x^2 - 3x =$$

$$4x^2 - 2x^2 =$$

$$2x^2 = 2x^2$$

Ta Da!

Steady and Transient States

For some linear equations, the term y_c decays as x (or t) grows. For example

$$\frac{dy}{dx} + y = 3xe^{-x} \quad \text{has solution} \quad y = \frac{3}{2}x^2e^{-x} + Ce^{-x}.$$

$$\text{Here,} \quad y_p = \frac{3}{2}x^2e^{-x} \quad \text{and} \quad y_c = Ce^{-x}.$$

Such a decaying complementary solution is called a **transient state.**

The corresponding particular solution is called a **steady state.**

Bernoulli Equations

Suppose $P(x)$ and $f(x)$ are continuous on some interval (a, b) and n is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is called a **Bernoulli** equation.

Observation: This equation has the flavor of a linear ODE, but since $n \neq 0, 1$ it is necessarily nonlinear. So our previous approach involving an integrating factor does not apply directly. Fortunately, we can use a change of variables to obtain a related linear equation.