Section 3: Separation of Variables

The simplest type of differential equation we could encounter would be of the form

\[ \frac{dy}{dx} = g(x). \]

No special techniques are needed; we just integrate

\[ \int \frac{dy}{dx} \, dx = \int g(x) \, dx \]

\[ \Rightarrow \quad \int dy = \int g(x) \, dx \]

\[ y = G(x) + C \]

where \( G'(x) = g(x) \)
Separable Equations

**Definition:** The first order equation \( y' = f(x, y) \) is said to be **separable** if the right side has the form

\[
f(x, y) = g(x)h(y).
\]

That is, a separable equation is one that has the form

\[
\frac{dy}{dx} = g(x)h(y).
\]
Determine which (if any) of the following are separable.

(a) \( \frac{dy}{dx} = x^3 y \) is separable \( g(x) = x^3 \), \( h(y) = y \)

(b) \( \frac{dy}{dx} = 2x + y \) is not separable
(c) \[ \frac{dy}{dx} = \sin(xy^2) \] is not separable

(d) \[ \frac{dy}{dt} - te^{t-y} = 0 \]  
\[ \Rightarrow \frac{dy}{dt} = te^{t-y} = te^t \cdot e^{-y} \]

This is separable with \( g(t) = te^t \) and \( h(y) = e^{-y} \).
Solving Separable Equations

Let’s assume that it’s safe to divide by \( h(y) \) and let’s set \( p(y) = 1/h(y) \). We solve (usually find an implicit solution) by separating the variables.

\[
\frac{dy}{dx} = g(x)h(y) \\
\Rightarrow \quad \frac{1}{h(y)} \frac{dy}{dx} = g(x) \\
\Rightarrow \quad p(y) \frac{dy}{dx} = g(x)dx \\
\Rightarrow \quad \int p(y)dy = \int g(x)dx \\
\Rightarrow \quad P(y) = G(x) + C
\]

an implicit family of solutions.
Solve the ODE

\[
\frac{dy}{dx} = -\frac{x}{y} 
\]

separately

\[
y \frac{dy}{dx} = -x \quad \Rightarrow \quad \int y \, dy = -\int x \, dx \quad \Rightarrow \quad \frac{1}{2} y^2 = -\frac{1}{2} x^2 + C
\]

Let \( k = 2C \) we get \( y^2 = -x^2 + k \)

\[
x^2 + y^2 = k \quad \text{a one parameter family of solutions}
\]
An IVP\(^1\)

\(te^{t-y} \, dt - dy = 0, \quad y(0) = 1\)

Let's separate variables

\[dy = te^{t-y} \, dt = te^t e^{-y} \, dt\]

\[\frac{1}{e^{-y}} dy = te^t \, dt\]

\[\int e^y \, dy = \int te^t \, dt\]

\[e^y = te^t - \int e^t \, dt\]

\(^1\)Recall IVP stands for \textit{initial value problem}.
\[ e^5 = te^t - e^t + C \]

Those are the solutions to the ODE.

Apply \( y(0) = 1 \).

\[ e^1 = 0 \cdot e^0 - e^0 + C \Rightarrow e = -1 + C \Rightarrow C = e + 1 \]

The solution to the IVP is given by

\[ y^* = te^t - e^t + e + 1 \]
Caveat regarding division by $h(y)$.

Recall that the IVP \( \frac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0 \)
has two solutions

\[ y(x) = \frac{x^4}{16} \quad \text{and} \quad y(x) = 0. \]

If we separate the variables

\[ \frac{1}{\sqrt{y}} \ dy = x \ dx \]

we lose the second solution.

**Why?** Dividing by $\sqrt{y}$ assumes $y \neq 0$. 

Caveat regarding division by $h(y)$.

We can look for solutions that may be lost using separation of variables. Suppose $y_0$ is such that $h(y_0) = 0$. Then $y = y_0$ is a constant solution of

$$\frac{dy}{dx} = g(x)h(y) \quad \text{subject to} \quad y(x_0) = y_0$$

To see that $y(x) = y_0$ solves the IVP, note $y(x_0) = y_0$. And since $h(y_0) = 0$,

$$\frac{dy}{dx} = \frac{d}{dx} y_0 = 0 = g(x) h(y_0) = g(x) \cdot 0 = 0$$

Hence the constant function solves the ODE and the initial condition.
Section 4: First Order Equations: Linear

A first order linear equation has the form

\[ a_1(x) \frac{dy}{dx} + a_0(x)y = g(x). \]

If \( g(x) = 0 \) the equation is called **homogeneous**. Otherwise it is called **nonhomogeneous**.

Provided \( a_1(x) \neq 0 \) on the interval \( I \) of definition of a solution, we can write the **standard form** of the equation

\[ \frac{dy}{dx} + P(x)y = f(x). \]

We’ll be interested in equations (and intervals \( I \)) for which \( P \) and \( f \) are continuous on \( I \).
Solutions (the General Solution)

\[
\frac{dy}{dx} + P(x)y = f(x).
\]

It turns out the solution will always have a basic form of \( y = y_c + y_p \) where

- \( y_c \) is called the **complementary** solution and would solve the problem
  \[
  \frac{dy}{dx} + P(x)y = 0
  \]
  (called the associated homogeneous equation), and
- \( y_p \) is called the **particular** solution, and is heavily influenced by the function \( f(x) \).

The cool thing is that our solution method will get both parts in one process—we won’t get this benefit with higher order equations!
Motivating Example

\[ x^2 \frac{dy}{dx} + 2xy = e^x \]

1st order linear, not in standard form.

The left side looks like a product rule, the derivative of one product. Note

\[ \frac{d}{dx} (x^2 y) = x^2 \frac{dy}{dx} + 2xy \quad \text{our left side.} \]

The ODE is \( \frac{d}{dx} (x^2 y) = e^x \)

We can solve, i.e. isolate \( y \), by integrating...
\[ \int \frac{d}{dx} \left( x^2 y \right) \, dx = \int e^x \, dx \]

\[ x^2 y = e^x + C \]

Now isolate \( y \) with division

\[ y = \frac{e^x + C}{x^2} \]

This can be written as

\[ y = \frac{e^x}{x^2} + \frac{C}{x^2} \]

This is NOT

\[ y = \frac{e^x}{x^2} + C \]
Derivation of Solution via Integrating Factor

Solve the equation in standard form

\[
\frac{dy}{dx} + P(x)y = f(x)
\]

We'll multiply our equation by a function \( \mu(x) \) in such a way that the resulting left side is a product rule. Let's assume \( \mu(x) > 0 \).

\[
\mu(x) \frac{dy}{dx} + \mu(x) P(x)y = \mu(x) f(x)
\]

Want this to be the derivative of a product. \( \frac{d}{dx} (\mu y) \).
Compare: \[
\frac{d}{dx} (\mu y) = \mu \frac{dy}{dx} + \frac{d\mu}{dx} y
\]

Matching requires
\[
\frac{d\mu}{dx} y = \mu P(x) y
\]

Cancel the \( y \)
\[
\frac{d\mu}{dx} = \mu P(x) \quad \text{a separable eqn in } \mu.
\]

Separate
\[
\frac{1}{\mu} \frac{d\mu}{dx} = P(x)
\]
\[
\int \frac{1}{\mu} d\mu = \int P(x) \, dx
\]
\ln \mu = \int p(x) \, dx \Rightarrow \mu = e^{\int p(x) \, dx}

This is called an integrating factor.

The equation becomes

\[ \frac{d}{dx} \left[ \mu(x)y \right] = \mu(x)f(x) \]

We'll integrate both sides to find \( y \).

(We'll finish this next time.)