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Section 3: Separation of Variables

The simplest type of differential equation we could encounter would be of the form

$$\frac{dy}{dx} = g(x).$$

No special techniques are needed; we just integrate

$$\int \underbrace{\frac{dy}{dx}\,dx}_{=\,dy} = \int g(x)\,dx \qquad \Rightarrow \qquad \int dy = \int g(x)\,dx$$

$$y = G(x) + C$$

where $\quad G'(x) = g(x)$

Separable Equations

Definition: The first order equation $y' = f(x, y)$ is said to be **separable** if the right side has the form

$$f(x, y) = g(x)h(y).$$

That is, a separable equation is one that has the form

$$\frac{dy}{dx} = g(x)h(y).$$

Determine which (if any) of the following are separable.

(a) $\frac{dy}{dx} = x^3 y$ Is separable $g(x) = x^3$, $h(y) = y$

(b) $\frac{dy}{dx} = 2x + y$ Is not separable

(c) $\frac{dy}{dx} = \sin(xy^2)$ Is not separable

(d) $\frac{dy}{dt} - te^{t-y} = 0$ $\Rightarrow$ $\frac{dy}{dt} = te^{t-y} = te^{t} \cdot e^{-y}$

This is separable w/ $g(t) = te^{t}$, $h(y) = e^{-y}$

Solving Separable Equations

Let's assume that it's safe to divide by $h(y)$ and let's set $p(y) = 1/h(y)$. We solve (usually find an implicit solution) by **separating the variables.**

$$\frac{dy}{dx} = g(x)h(y) \quad \overset{\text{divide by } h}{\Longrightarrow} \quad \frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

$$\Rightarrow \quad p(y)\frac{dy}{dx} = g(x) \quad \overset{\text{mult. by } dx}{\Longrightarrow} \quad p(y)\frac{dy}{dx}\,dx = g(x)\,dx$$

Using $\frac{dy}{dx}dx = dy$ $\quad p(y)\,dy = g(x)\,dx$ (variables seperated)

$$\int p(y)\,dy = \int g(x)\,dx \quad \Rightarrow \quad \boxed{P(y) = G(x) + C} \qquad P'(y) = p(y),\ G'(x) = g(x)$$

an implicit family of solutions

Solve the ODE

separate

$$\frac{dy}{dx} = -\frac{x}{y} = -x\left(\frac{1}{y}\right) \implies y\frac{dy}{dx} = -x \Rightarrow y\frac{dy}{dx}dx = -x\,dx$$

$$\int y\,dy = -\int x\,dx \Rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$$

Let $k = 2C$ we get $y^2 = -x^2 + k$

$\boxed{x^2 + y^2 = k}$ a one parameter family of solutions

An IVP[1]

$$te^{t-y}\,dt - dy = 0, \qquad y(0) = 1$$

Let's separate variables

$$dy = te^{t-y}\,dt = te^{t}e^{-y}\,dt$$

$$\frac{1}{e^{-y}}\,dy = te^{t}\,dt$$

$$\int e^{y}\,dy = \int te^{t}\,dt$$

$$e^{y} = te^{t} - \int e^{t}\,dt$$

Int. by parts

$u = t \qquad du = dt$

$v = e^{t} \qquad dv = e^{t}\,dt$

[1] Recall IVP stands for *initial value problem*.

$$e^y = te^t - e^t + C$$

Those are the solutions to the ODE.

Apply $y(0)=1$.

$$e^1 = 0\cdot e^0 - e^0 + C \Rightarrow e = -1 + C \Rightarrow C = e+1$$

The solution to the IVP is given by

$$\boxed{e^y = te^t - e^t + e + 1}$$

Caveat regarding division by $h(y)$.

Recalll that the IVP $\quad \dfrac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0$

has two solutions

$$y(x) = \frac{x^4}{16} \quad \text{and} \quad y(x) = 0.$$

If we separate the variables

$$\frac{1}{\sqrt{y}}\,dy = x\,dx$$

we lose the second solution.

Why? Dividing by $\sqrt{y}$ assumes $y \neq 0$.

Caveat regarding division by $h(y)$.

We can look for solutions that may be lost using separation of variables. Suppose y_0 is such that $h(y_0) = 0$. Then $y = y_0$ is a constant solution of

$$\frac{dy}{dx} = g(x)h(y) \quad \text{subject to} \quad y(x_0) = y_0$$

To see that $y(x) = y_0$ solves the IVP, note

$y(x_0) = y_0$. And since $h(y_0) = 0$,

$$\frac{dy}{dx} = \frac{d}{dx} y_0 = 0 = g(x)\, h(y_0) = g(x) \cdot 0 = 0$$

Hence the constant function solves the ODE and the initial condition.

Section 4: First Order Equations: Linear

A first order linear equation has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$ the equation is called **homogeneous.** Otherwise it is called **nonhomogeneous.**

Provided $a_1(x) \neq 0$ on the interval I of definition of a solution, we can write the **standard form** of the equation

$$P(x) = \frac{a_0(x)}{a_1(x)}$$

$$\frac{dy}{dx} + P(x)y = f(x). \qquad f(x) = \frac{g(x)}{a_1(x)}$$

We'll be interested in equations (and intervals I) for which P and f are continuous on I.

Solutions (the General Solution)

$$\frac{dy}{dx} + P(x)y = f(x).$$

It turns out the solution will always have a basic form of $y = y_c + y_p$ where

- y_c is called the **complementary** solution and would solve the problem
$$\frac{dy}{dx} + P(x)y = 0$$
(called the associated homogeneous equation), and
- y_p is called the **particular** solution, and is heavily influenced by the function $f(x)$.

The cool thing is that our solution method will get both parts in one process—we won't get this benefit with higher order equations!

Motivating Example

$$x^2\frac{dy}{dx}+2xy = e^x$$

1st order linear, not in standard form.

The left side looks like a product rule, the derivative of one product. Note

$$\frac{d}{dx}\left(x^2y\right) = x^2\frac{dy}{dx} + 2xy$$ our left side.

The ODE is $$\frac{d}{dx}\left(x^2y\right) = e^x$$

We can solve, i.e. isolate y, by integrating

$$\int \frac{d}{dx}(x^2 y)\,dx = \int e^x\,dx$$

$$x^2 y = e^x + C$$

Now isolate y with division

$$y = \frac{e^x + C}{x^2}$$

This can be written as $y = \dfrac{e^x}{x^2} + \dfrac{C}{x^2}$

This is NOT $y = \dfrac{e^x}{x^2} + C$

Derivation of Solution via Integrating Factor

Solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

we'll multiply our equation by a function $\mu(x)$ in such a way that the resulting left side is a product rule. Let's assume $\mu(x) > 0$.

$$\underbrace{\mu(x)\frac{dy}{dx} + \mu(x)P(x)y}_{\text{want this to be the derivative of a product}} = \mu(x)f(x)$$

want this to be the derivative of a product. $\frac{d}{dx}(\mu y)$

Compare: $\frac{d}{dx}(\mu y) = \mu \frac{dy}{dx} + \frac{d\mu}{dx} y$

Matching requires $\frac{d\mu}{dx} y = \mu P(x) y$

Cancel the y

$\frac{d\mu}{dx} = \mu P(x)$ a separable eqn in μ.

Separate $\frac{1}{\mu}\frac{d\mu}{dx} = P(x)$

$$\int \frac{1}{\mu}\, d\mu = \int P(x)\, dx$$

$$\ln \mu = \int P(x)\,dx \Rightarrow \mu = e^{\int P(x)\,dx}$$

This is called an integrating factor.

The equation becomes

$$\frac{d}{dx}\left[\mu(x)y\right] = \mu(x)f(x).$$

We'll integrate both sides to find y.

(we'll finish this next time.)