

Section 3: Separation of Variables

The simplest type of differential equation we could encounter would be of the form

$$\frac{dy}{dx} = g(x).$$

No special techniques are needed; we just integrate

$$\int \underbrace{\frac{dy}{dx}}_{dy} dx = \int g(x) dx \quad \Rightarrow \quad \int dy = \int g(x) dx$$
$$y = G(x) + C$$

where G is any antiderivative of $g(x)$.

Separable Equations

Definition: The first order equation $y' = f(x, y)$ is said to be **separable** if the right side has the form

$$f(x, y) = g(x)h(y).$$

That is, a separable equation is one that has the form

$$\frac{dy}{dx} = g(x)h(y).$$

product of fact. of x alone
and fact of y -alone.

Determine which (if any) of the following are separable.

(a) $\frac{dy}{dx} = x^3 y$

yes, it's separable $g(x) = x^3$, $h(y) = y$

(b) $\frac{dy}{dx} = 2x + y$

this is not separable

(c) $\frac{dy}{dx} = \sin(xy^2)$

No, it's not separable

(d) $\frac{dy}{dt} - te^{t-y} = 0 \Rightarrow \frac{dy}{dt} = te^{t-y} = te^t \cdot e^{-y}$

yes it's separable $g(t) = te^t$, $h(y) = e^{-y}$

Solving Separable Equations

Let's assume that it's safe to divide by $h(y)$ and let's set $p(y) = 1/h(y)$. We solve (usually find an implicit solution) by **separating the variables**.

$$\frac{dy}{dx} = g(x)h(y) \quad \xRightarrow{\text{divide by } h(y)} \quad \frac{1}{h(y)} \frac{dy}{dx} = g(x) \quad p = \frac{1}{h}$$

$$p(y) \frac{dy}{dx} = g(x) \quad \xRightarrow{\text{mult. by } dx} \quad p(y) \underbrace{\frac{dy}{dx} dx}_{dy} = g(x) dx$$

$$p(y) dy = g(x) dx \quad \text{variables are separated}$$

$$\int p(y) dy = \int g(x) dx$$

$\Rightarrow P(y) = G(x) + C$
1-parameter family
of solutions

$$P'(y) = p(y)$$

$$G'(x) = g(x)$$

Solve the ODE

$$\frac{dy}{dx} = -\frac{x}{y} = -x \cdot \frac{1}{y} \quad \Rightarrow \quad y \frac{dy}{dx} = -x$$

$$\Rightarrow y \frac{dy}{dx} dx = -x dx \quad \Rightarrow \quad \int y dy = \int -x dx$$

$$\frac{1}{2} y^2 = -\frac{1}{2} x^2 + C$$

Letting $k = 2C$

$$y^2 = -x^2 + k \quad \Rightarrow \quad x^2 + y^2 = k$$

An IVP¹

$$te^{t-y} dt - dy = 0, \quad y(0) = 1$$

Rearrange

$$dy = te^{t-y} dt = te^t \cdot e^{-y} dt$$

$$\frac{1}{e^{-y}} dy = te^t dt$$

$$\int e^y dy = \int te^t dt$$

$$e^y = te^t - \int e^t dt$$

Int. by parts

$$u = t \quad du = dt$$

$$v = e^t \quad dv = e^t dt$$

¹Recall IVP stands for *initial value problem*.

$$e^y = te^t - e^t + C$$

1-parameter family
of solutions
to the ODE.

Impose the condition $y(0)=1$. $y=1$ when $t=0$

$$e^1 = 0 \cdot e^0 - e^0 + C \Rightarrow e = -1 + C \Rightarrow C = e + 1$$

The solution the IVP is given implicitly by

$$e^y = te^t - e^t + e + 1$$

Caveat regarding division by $h(y)$.

Recall that the IVP $\frac{dy}{dx} = x\sqrt{y}$, $y(0) = 0$

has two solutions

$$y(x) = \frac{x^4}{16} \quad \text{and} \quad y(x) = 0.$$

If we separate the variables

$$\frac{1}{\sqrt{y}} dy = x dx$$

we lose the second solution.

Why? When dividing by \sqrt{y} , we assume that $y \neq 0$.

Caveat regarding division by $h(y)$.

We can look for solutions that may be lost using separation of variables. Suppose y_0 is such that $h(y_0) = 0$. Then $y = y_0$ is a constant solution of

$$\frac{dy}{dx} = g(x)h(y) \quad \text{subject to} \quad y(x_0) = y_0$$

If $h(y_0) = 0$, and $y(x) = y_0$ then

$$\frac{d}{dx} y = \frac{d}{dx} y_0 = 0 \quad \frac{dy}{dx} = 0 = g(x)h(y_0) = g(x) \cdot 0 = 0$$

And if $y(x) = y_0$ then $y(x_0) = y_0$.

Section 4: First Order Equations: Linear

A first order linear equation has the form

$$a_1(x) \frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$ the equation is called **homogeneous**. Otherwise it is called **nonhomogeneous**.

Provided $a_1(x) \neq 0$ on the interval I of definition of a solution, we can write the **standard form** of the equation

$$\frac{dy}{dx} + P(x)y = f(x).$$

$$P(x) = \frac{a_0(x)}{a_1(x)}$$

$$f(x) = \frac{g(x)}{a_1(x)}$$

We'll be interested in equations (and intervals I) for which P and f are continuous on I .

Solutions (the General Solution)

$$\frac{dy}{dx} + P(x)y = f(x).$$

It turns out the solution will always have a basic form of $y = y_c + y_p$ where

- ▶ y_c is called the **complementary** solution and would solve the problem

$$\frac{dy}{dx} + P(x)y = 0$$

(called the associated homogeneous equation), and

- ▶ y_p is called the **particular** solution, and is heavily influenced by the function $f(x)$.

The cool thing is that our solution method will get both parts in one process—we won't get this benefit with higher order equations!

Motivating Example

$$x^2 \frac{dy}{dx} + 2xy = e^x$$

Not in standard form, but we'll work with it like this.

Note, the left side is $\frac{d}{dx}(x^2 y)$, the derivative of one product.

$$\frac{d}{dx}(x^2 y) = x^2 \frac{dy}{dx} + 2xy$$

So our ODE is

$$\frac{d}{dx}(x^2 y) = e^x$$

To find y , we integrate and divide. We'll back out y

Integrate $\int \frac{d}{dx} (x^2 y) dx = \int e^x dx$

$$x^2 y = e^x + C$$

Solve for y $y = \frac{e^x + C}{x^2}$ ← this is the solution family

This can be expressed as $y = \frac{e^x}{x^2} + \frac{C}{x^2}$

This is NOT $y = \frac{e^x}{x^2} + C$