January 23 Math 2306 sec. 60 Spring 2019

Section 4: First Order Equations: Linear

We wish to solve a first order linear equation. Such an equation in standard form looks like

$$\frac{dy}{dx} + P(x)y = f(x).$$

We obtain a **general solution** in the form $y = y_c + y_p$ using an integrating factor.

General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx}f(x)\,dx + C\right)$$

Solve the ODE

$$\frac{dy}{dx} + y = 3xe^{-x}$$

This is in standard form with

$P(x) = 1$

Integrating factor $\mu = e^{\int P(x)\,dx} = e^{\int dx} = e^{x}$

* In $\int P(x)\,dx$, we can take the constant of integration to be anything. I set it to zero.

Multiply the ODE by μ

$$e^{x}\frac{dy}{dx} + e^{x}y = e^{x}\left(3xe^{-x}\right) = 3x\,e^{x}e^{-x} = 3x$$

$$\frac{d}{dx}\left(e^{x}y\right) = 3x$$

Integrate $$\int \frac{d}{dx}\left(e^{x}y\right)dx = \int 3x\,dx$$

$$e^{x}y = \frac{3}{2}x^2 + C$$

(This step required the constant of integration.)

Divide by μ to get the solution y.

$$y = \frac{\frac{3}{2}x^2 + C}{e^{x}}$$

$$\boxed{y = \frac{3}{2}x^2e^{-x} + Ce^{-x}}$$

$y_c = Ce^{-x}$ and

$y_p = \frac{3}{2}x^2e^{-x}$

Solve the IVP

$$x\frac{dy}{dx}-y=2x^2,\ x>0\quad y(1)=5$$

Put the ODE in standard form (divide by x)

$$\frac{dy}{dx}-\frac{1}{x}y=\frac{2x^2}{x}=2x$$

$$P(x)=\frac{-1}{x}$$

Build the integrating factor $\mu=e^{\int P(x)\,dx}$

$$\mu=e^{\int\frac{-1}{x}dx}=e^{-\ln x}=e^{\ln x^{-1}}=x^{-1}$$

$$x^{-1}\frac{dy}{dx} - x^{-1}\left(\frac{1}{x}\right)y = x^{-1}(2x)$$

$$\frac{d}{dx}\left(x^{-1}y\right) = 2$$

$$\int \frac{d}{dx}\left(x^{-1}y\right)dx = \int 2\,dx$$

$$x^{-1}y = 2x + C$$

$$y = \frac{2x+C}{x^{-1}} = 2x^2 + Cx$$

The general solution to the ODE is

$$y = 2x^2 + Cx$$

Now impose the initial condition $y(1) = 5$

$$5 = 2(1^2) + C(1) = 2 + C$$

$$\Rightarrow \quad C = 3$$

The solution to the IVP is

$$y = 2x^2 + 3x$$

Verify

Just for giggles, lets verify that our solution $y = 2x^2 + 3x$ really does solve the differential equation we started with

$$x\frac{dy}{dx} - y = 2x^2.$$

we'll substitute: $y = 2x^2 + 3x$, $y' = 4x + 3$

$$x(4x+3) - (2x^2 + 3x) =$$

$$4x^2 + 3x - 2x^2 - 3x =$$

$$4x^2 - 2x^2 =$$

$$2x^2 = 2x^2$$

Ta Da!

Steady and Transient States

For some linear equations, the term y_c decays as x (or t) grows. For example

$$\frac{dy}{dx} + y = 3xe^{-x} \quad \text{has solution} \quad y = \frac{3}{2}x^2e^{-x} + Ce^{-x}.$$

$$\text{Here,} \quad y_p = \frac{3}{2}x^2e^{-x} \quad \text{and} \quad y_c = Ce^{-x}.$$

Such a decaying complementary solution is called a **transient state.**

The corresponding particular solution is called a **steady state.**

Bernoulli Equations

Suppose $P(x)$ and $f(x)$ are continuous on some interval (a, b) and n is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is called a **Bernoulli** equation.

Observation: This equation has the flavor of a linear ODE, but since $n \neq 0, 1$ it is necessarily nonlinear. So our previous approach involving an integrating factor does not apply directly. Fortunately, we can use a change of variables to obtain a related linear equation.