Section 1.2: Row Reduction and Echelon Forms

A few things to recall:

- Row equivalent matrices correspond to equivalent systems.
- The rref for a matrix is unique.
- The pivot positions and pivot columns correspond to the locations of the leading ones in an rref.
Echelon Form & Solving a System

Consider the reduced echelon matrix. Identify the pivot positions. Then, describe the solution set for the system of equations whose augmented matrix is row equivalent.

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 3 \\
0 & 0 & 1 & 0 & -2 & 4 \\
0 & 0 & 0 & 1 & 0 & -9 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Leading ones, pivot positions in columns 1, 3, and 4.

The system reads as

\[
\begin{align*}
x_1 + x_2 &= 3 \\
x_3 - 2x_5 &= 4 \\
x_4 &= -9 \\
0 &= 0 \quad \text{always true}
\end{align*}
\]
We can express the solution set as a list

\[ x_1 = 3 - x_2 \]
\[ x_3 = 1 + 2x_5 \]
\[ x_4 = -9 \]

\( x_2, x_5 \) are free

By "free" we mean they can take on any real number value.
Choosing $x_2$ and $x_5$ as free (as opposed to $x_1$ and $x_3$) follows the convention that free variables correspond to non-pivot columns.

Variables $x_1$, $x_3$, and $x_4$ are called basic variables.

Basic variable ↔ pivot column

Free variables ↔ non-pivot column
We can express basic variables in terms of free variables.

We won't express basics in terms of other basics, free in terms of free or free in terms of basic.
Consistent versus Inconsistent Systems

Consider each rref. Determine if the underlying system (the one with this as its augmented matrix) is consistent or inconsistent.

\[
\begin{bmatrix}
1 & 2 & 0 & 0 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & -3
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 & 2 & 3 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

- **First System:**
  - \(x_1 = -2x_2\)
  - \(x_3 = 4\)
  - \(x_2\) is free
  - Consistent
  - Many solutions

- **Second System:**
  - \(x_1 = 0\)
  - \(x_2 = 4\)
  - \(x_3 = -3\)
  - Consistent
  - One solution

- **Third System:**
  - Last eqn reads as \(0 = 1\)
  - Always false
  - Inconsistent
An Existence and Uniqueness Theorem

**Theorem:** A linear system is consistent if and only if the right most column of the augmented matrix is **NOT** a pivot column. That is, if and only if each echelon form **DOES NOT** have a row of the form

\[
\begin{bmatrix}
0 & 0 & \cdots & 0 & b
\end{bmatrix}, \quad \text{for some nonzero } b.
\]

If a linear system is consistent, then it has

(i) exactly one solution if there are no free variables, or

(ii) infinitely many solutions if there is at least one free variable.
Section 1.3: Vector Equations

**Definition:** A matrix that consists of one column is called a **column vector** or simply a **vector**.

In print, vectors are denoted in bold face. In hand writing, we use an over mark, usually an arrow, e.g. \( \vec{u} \) or \( \vec{v} \). The numbers in a vector are called components or entries.

The set of vectors of the form \( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) with \( x_1 \) and \( x_2 \) any real numbers is denoted by \( \mathbb{R}^2 \) (read ”R two”). It’s the set of all real ordered pairs.

The standard Cartesian plane.
Geometry

Each vector \([ x_1 \\ x_2 ]\) corresponds to a point in the Cartesian plane. We can equate them with ordered pairs written in the traditional format
\[
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
= (x_1, x_2).
\]
This is not to be confused with a row matrix.

We can identify vectors with points or with directed line segments emanating from the origin (little arrows).
Figure: Vectors characterized as points, and vectors characterized as directed line segments.
Algebraic Operations

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, and $c$ be a scalar\(^1\).

**Scalar Multiplication:** The scalar multiple of $\mathbf{u}$

$$c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}.$$ Componentwise

**Vector Addition:** The sum of vectors $\mathbf{u}$ and $\mathbf{v}$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}.$$

**Vector Equivalence:** Equality of vectors is defined by

$$\mathbf{u} = \mathbf{v} \quad \text{if and only if} \quad u_1 = v_1 \quad \text{and} \quad u_2 = v_2.$$ \[1\]

\(^{1}\)A scalar is an element of the set from which $u_1$ and $u_2$ come. For our purposes, a scalar is a real number.
Examples

\[ \mathbf{u} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} \]

Evaluate

(a) \(-2\mathbf{u}\)

\[ -2 \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \cdot 4 \\ -2 \cdot (-2) \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix} \]

\[ 3\mathbf{v} = 3 \begin{bmatrix} -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \cdot (-1) \\ 3 \cdot 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 21 \end{bmatrix} \]

(b) \(-2\mathbf{u} + 3\mathbf{v}\)

\[ \begin{bmatrix} -8 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 21 \end{bmatrix} = \begin{bmatrix} -8 + (-3) \\ 4 + 21 \end{bmatrix} = \begin{bmatrix} -11 \\ 25 \end{bmatrix} \]
Examples

\[ \mathbf{u} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \]

Is it true that \( \mathbf{w} = -\frac{3}{4} \mathbf{u} \)?

\[
-\frac{3}{4} \mathbf{u} = \frac{3}{4} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \cdot 4 \\ \frac{3}{4} \cdot (-2) \end{bmatrix} = \begin{bmatrix} \frac{-3}{2} \end{bmatrix} = \mathbf{w}
\]

Since both corresponding entries are equal. So yes \( -\frac{3}{4} \mathbf{u} = \mathbf{w} \).
Geometry of Algebra with Vectors

Figure: Left: \( \frac{1}{2}(-4, 1) = (-2, 1/2) \). Right: \((-4, 1) + (2, 5) = (-2, 6)\)
Geometry of Algebra with Vectors

Scalar Multiplication: stretches or compresses a vector but can only change direction by an angle of 0 (if $c > 0$) or $\pi$ (if $c < 0$). We’ll see that $0\vec{u} = (0, 0)$ for any vector $\vec{u}$. 

$0\vec{u} = (0, 0)$

3 times as long

twice the length

opposite orientation
Vector Addition: The sum $\mathbf{u} + \mathbf{v}$ of two vectors (each different from $(0,0)$) is the fourth vertex of a parallelogram whose other three vertices are $(u_1, u_2)$, $(v_1, v_2)$, and $(0,0)$.
Vectors in $\mathbb{R}^n$

A vector in $\mathbb{R}^3$ is a $3 \times 1$ column matrix. These are ordered triples. For example

$$a = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \quad \text{or} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$  

A vector in $\mathbb{R}^n$ for $n \geq 2$ is a $n \times 1$ column matrix. These are ordered $n$-tuples. For example

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = (x_1, x_2, \ldots, x_n).$$

The Zero Vector: is the vector whose entries are all zeros. It will be denoted by $\mathbf{0}$ or $\vec{0}$ and is not to be confused with the scalar $0$. 
Algebraic Properties on $\mathbb{R}^n$

For every $u$, $v$, and $w$ in $\mathbb{R}^n$ and scalars $c$ and $d$:

(i) $u + v = v + u$

(ii) $(u + v) + w = u + (v + w)$

(iii) $u + 0 = 0 + u = u$

(iv) $u + (-u) = -u + u = 0$

(v) $c(u + v) = cu + cv$

(vi) $(c + d)u = cu + du$

(vii) $c(du) = d(cu) = (cd)u$

(viii) $1u = u$

\[2\text{The term } -u \text{ denotes } (-1)u.\]
Definition: Linear Combination

A linear combination of vectors $v_1, \ldots, v_p$ in $\mathbb{R}^n$ is a vector $y$ of the form

$$y = c_1 v_1 + \cdots + c_p v_p$$

where the scalars $c_1, \ldots, c_p$ are often called weights.

For example, suppose we have two vectors $v_1$ and $v_2$. Some linear combinations include

$$3v_1, \quad -2v_1 + 4v_2, \quad \frac{1}{3}v_2 + \sqrt{2}v_1, \quad \text{and} \quad 0 = 0v_1 + 0v_2.$$
Example

Let \( a_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \ a_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \) and \( b = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix} \). Determine if \( b \) can be written as a linear combination of \( a_1 \) and \( a_2 \).

We can state this possibility as an equation:

Are there scalars \( c_1, c_2 \) such that

\[
\begin{align*}
&c_1 \vec{a}_1 + c_2 \vec{a}_2 = \vec{b} \\
\Rightarrow & c_1 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}
\end{align*}
\]

By our operations

\[
\begin{align*}
&\begin{bmatrix} c_1 \\ -2c_1 \\ -c_1 \end{bmatrix} + \begin{bmatrix} 3c_2 \\ 0 \\ 2c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix} \\
\Rightarrow & \begin{bmatrix} c_1 + 3c_2 \\ -2c_1 \\ -c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}
\end{align*}
\]
By vector equivalence, this would require

\[ c_1 + 3c_2 = -2 \]
\[ -2c_1 = -2 \]
\[ -c_1 + 2c_2 = -3 \]

A linear system of equations in 2 variables

Equation 2 requires \( c_1 = 1 \), then \( c_2 = \frac{1}{3} (-2-1) = -1 \)

and \( c_2 = \frac{1}{2} (-3+1) = -1 \).

Note that we could consider the augmented matrix

\[
\begin{bmatrix}
1 & 3 & -2 \\
-2 & 0 & -2 \\
-1 & 2 & -3
\end{bmatrix}
\]
The ref is \[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{bmatrix}
\]
The system is consistent, column 3 is not a pivot column.

There are two pivot columns and 2 variables, hence exactly one solution \( c_1 = 1, c_2 = -1 \).

Finally, yes \( \overrightarrow{b} \) can be written as a linear combination of \( \overrightarrow{a}_1 \) and \( \overrightarrow{a}_2 \).
Some Convenient Notation

Letting $\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}$, and in general $\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$, for $j = 1, \ldots, n$, we can denote the $m \times n$ matrix whose columns are these vectors by

$$[\mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$  

Note that each vector $\mathbf{a}_j$ is a vector in $\mathbb{R}^m$. 
Vector and Matrix Equations

The vector equation

\[ x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = b \]

has the same solution set as the linear system whose augmented matrix is

\[ \begin{bmatrix} a_1 & a_2 & \cdots & a_n & b \end{bmatrix}. \]  \hspace{1cm} (1)

In particular, \( b \) is a linear combination of the vectors \( a_1, \ldots, a_n \) if and only if the linear system whose augmented matrix is given in (1) is consistent.
Definition of **Span**

Let $S = \{v_1, \ldots, v_p\}$ be a set of vectors in $\mathbb{R}^n$. The set of all linear combinations of $v_1, \ldots, v_p$ is denoted by

$$\text{Span}\{v_1, \ldots, v_p\} = \text{Span}(S).$$

It is called the **subset of $\mathbb{R}^n$ spanned by (a.k.a. generated by)** the set $\{v_1, \ldots, v_p\}$.

To say that a vector $b$ is in $\text{Span}\{v_1, \ldots, v_p\}$ means that there exists a set of scalars $c_1, \ldots, c_p$ such that $b$ can be written as

$$c_1v_1 + \cdots + c_pv_p.$$
Span: Equivalent Statements

If \( \mathbf{b} \) is in \( \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \), then \( \mathbf{b} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p \). From the previous result, we know this is equivalent to saying that the vector equation

\[
x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{b}
\]

has a solution. This is in turn the same thing as saying the linear system with augmented matrix

\[
[\mathbf{v}_1 \ \cdots \ \mathbf{v}_p \ \mathbf{b}]
\]

is consistent.
Examples

Let $a_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, and $a_2 = \begin{bmatrix} -1 \\ 4 \\ -2 \end{bmatrix}$.

(a) Determine if $b = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ is in $\text{Span}\{a_1, a_2\}$.

$b$ is in $\text{Span}\{a_1, a_2\}$ if the system $Ax = b$ augmented matrix $\begin{bmatrix} a_1 & a_2 & b \end{bmatrix}$ is consistent.

\[
\begin{bmatrix}
1 & -1 & 4 \\
1 & 4 & 2 \\
2 & -2 & 1 \\
\end{bmatrix}
\xrightarrow{\text{ref}}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\text{ (using TI-89)}
\]
The rightmost column is a pivot column.

The system is inconsistent.

\[ \vec{b} \text{ is not in } \text{span} \{ \vec{a}_1, \vec{a}_2 \} \]
(b) Determine if \( \mathbf{b} = \begin{bmatrix} 5 \\ -5 \\ 10 \end{bmatrix} \) is in \( \text{Span}\{\mathbf{a}_1, \mathbf{a}_2\} \).

\[
\begin{bmatrix}
\mathbf{a}_1, \mathbf{a}_2, \mathbf{b} \\
\end{bmatrix} = \begin{bmatrix} 1 & -1 & 5 \\ 1 & 4 & -5 \\ 2 & -2 & 10 \end{bmatrix}
\]

\[
\rightarrow \text{ref} \begin{bmatrix} 1 & 0 & 3 \\
0 & 1 & -2 \\
0 & 0 & 0 \end{bmatrix}
\]

We see that the system is consistent, moreover
\[ \mathbf{b} = 3 \mathbf{a}_1 - 2 \mathbf{a}_2. \]

Hence $\mathbf{b}$ is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$. 
Another Example

Give a geometric description of the subset of $\mathbb{R}^2$ given by $\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

Any vector in $\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ has the form $c \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix}$. This is the classic x-axis in $\mathbb{R}^2$. 