January 23 Math 3260 sec. 56 Spring 2018

Section 1.2: Row Reduction and Echelon Forms

A few things to recall:

- Row equivalent matrices correspond to equivalent systems.
- The rref for a matrix is unique.
- The pivot positions and pivot columns correspond to the locations of the leading ones in an rref.

Echelon Form & Solving a System

Consider the reduced echelon matrix. Identify the pivot positions. Then, describe the solution set for the system of equations whose augmented matrix is row equivalent.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 0 & 1 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot positions, pivot columns are columns 1, 3, and 4.

The system reads as

$$\begin{aligned} x_1 + x_2 \qquad\qquad &= 3 \\ x_3 \quad -2x_5 &= 4 \\ x_4 \qquad &= -9 \\ 0 &= 0 \leftarrow \text{always true} \end{aligned}$$

we can arrange the solutions in list format as

$x_1 = 3 - x_2$

$x_3 = 4 + 2x_5$

$x_4 = -9$

x_2 and x_5 can be any real number

We'll call x_2 and x_5 free variables.

We could write $x_2 = 3 - x_1$; however, we will follow the convention that free variables correspond to non-pivot columns.

The remaining variables are called basic variables, here x_1, x_3, x_4.

Basic variable $\leftrightarrow$ pivot column

free variables $\leftrightarrow$ non pivot column.

We will stick with the convention that basic variables may be expressed in terms of free variables.

We never express basics in terms of basics, free in terms of basics, or free in terms of frees.

Consistent versus Inconsistent Systems

Consider each rref. Determine if the underlying system (the one with this as its augmented matrix) is consistent or inconsistent.

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$x_1 = -2x_2$

$x_3 = 4$

x_2 - free

Consistent

∞ - many solutions

$x_1 = 0$

$x_2 = 4$

$x_3 = -3$

Consistent

Exactly one solution

Last row reads as $0 = 1$.

The system is inconsistent.

An Existence and Uniqueness Theorem

Theorem: A linear system is consistent if and only if the right most column of the augmented matrix is **NOT** a pivot column. That is, if and only if each echelon form **DOES NOT** have a row of the form

$$[0\ 0\ \cdots\ 0\ b], \quad \text{for some nonzero } b.$$

If a linear system is consistent, then it has

(i) exactly one solution if there are no free variables, or

(ii) infinitely many solutions if there is at least one free variable.

Section 1.3: Vector Equations

Definition: A matrix that consists of one column is called a **column vector** or simply a **vector.**

In print, vectors are denoted using bold face.

In handwriting, an arrow is placed over a vector.

e.g. $\vec{u}$, $\vec{v}$.

The set of vectors of the form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with x_1 and x_2 any real numbers is denoted by $\mathbb{R}^2$ (read "R two"). It's the set of all real ordered pairs.

The Cartesian plane.

Geometry

Each vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ corresponds to a point in the Cartesian plane. We can equate them with ordered pairs written in the traditional format $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (x_1, x_2)$. This is **not to be confused with a row matrix.**

note the parentheses and comma

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq [x_1 \;\; x_2]$$

no comma here

We can identify vectors with points or with directed line segments emanating from the origin (little arrows).

Geometry

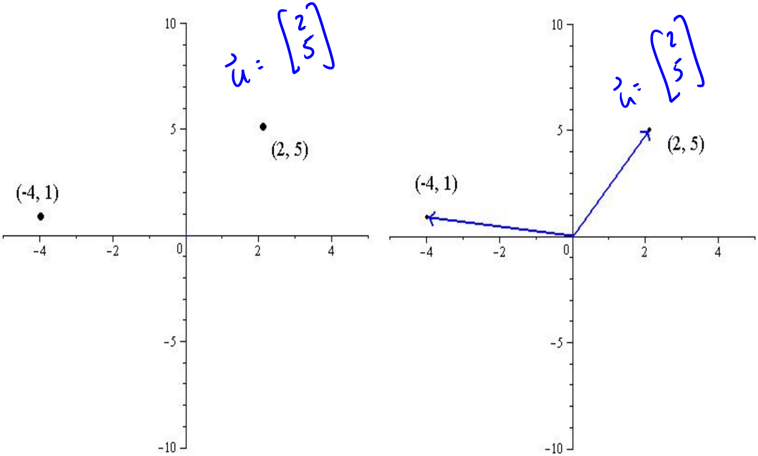


Figure: Vectors characterized as points, and vectors characterized as directed line segments.

Algebraic Operations

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, and c be a scalar[1].

The u_1, u_2, v_1, v_2 are called entries or components.

Scalar Multiplication: The scalar multiple of $\mathbf{u}$

$$c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}.$$

← component wise

Vector Addition: The sum of vectors $\mathbf{u}$ and $\mathbf{v}$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

Vector Equivalence: Equality of vectors is defined by

$$\mathbf{u} = \mathbf{v} \quad \text{if and only if} \quad u_1 = v_1 \quad \text{and} \quad u_2 = v_2.$$

[1] A **scalar** is an element of the set from which u_1 and u_2 come. For our purposes, a scalar is a *real* number.

Examples

$$\mathbf{u} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} -3 \\ \frac{3}{2} \end{bmatrix}$$

Evaluate

(a) $-2\mathbf{u}$ $= -2\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -2\cdot 4 \\ -2\cdot(-2) \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$

$3\vec{v} = 3\begin{bmatrix} -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 3(-1) \\ 3\cdot 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 21 \end{bmatrix}$

(b) $-2\mathbf{u}+3\mathbf{v}$

$= \begin{bmatrix} -8 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ 21 \end{bmatrix} = \begin{bmatrix} -8+(-3) \\ 4+21 \end{bmatrix} = \begin{bmatrix} -11 \\ 25 \end{bmatrix}$

Examples

$$\mathbf{u} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} -3 \\ \frac{3}{2} \end{bmatrix}$$

Is it true that $\mathbf{w} = -\frac{3}{4}\mathbf{u}$?

$$\frac{-3}{4}\vec{u} = \frac{-3}{4}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{-3}{4}(4) \\ \frac{-3}{4}(-2) \end{bmatrix} = \begin{bmatrix} -3 \\ \frac{3}{2} \end{bmatrix}$$

Each corresponding entry is equivalent, so

yes $\vec{w} = \frac{-3}{4}\vec{u}$.

Geometry of Algebra with Vectors

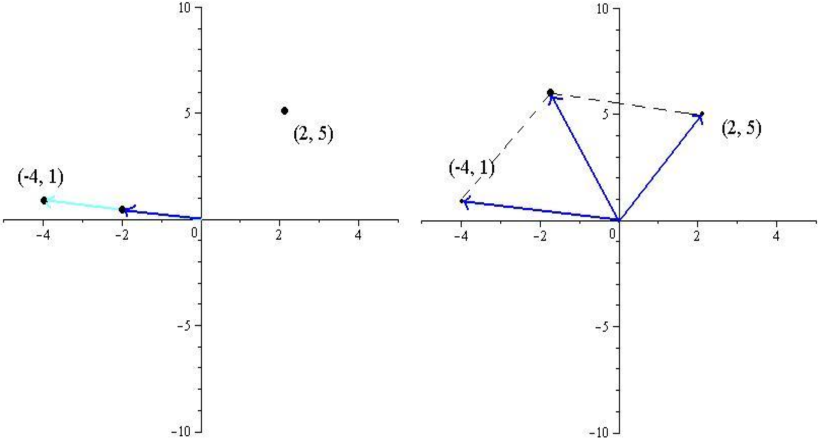
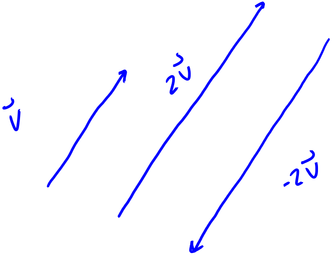


Figure: Left: $\frac{1}{2}(-4, 1) = (-2, 1/2)$. Right: $(-4, 1) + (2, 5) = (-2, 6)$

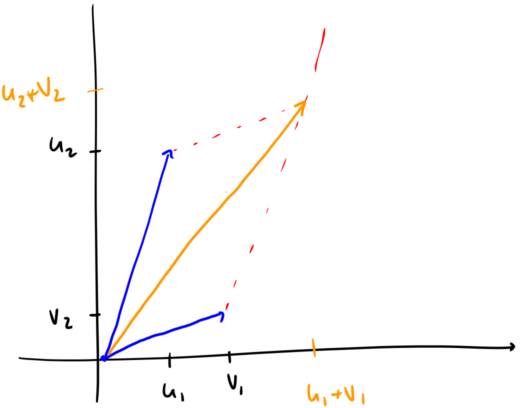
Geometry of Algebra with Vectors

Scalar Multiplication: stretches or compresses a vector but can only change direction by an angle of 0 (if $c > 0$) or π (if $c < 0$). We'll see that $0\mathbf{u} = (0, 0)$ for any vector $\mathbf{u}$.



Geometry of Algebra with Vectors

Vector Addition: The sum $\mathbf{u} + \mathbf{v}$ of two vectors (each different from $(0,0)$) is the the fourth vertex of a parallelogram whose other three vertices are (u_1, u_2), (v_1, v_2), and $(0,0)$.



Vectors in $\mathbb{R}^n$

A vector in $\mathbb{R}^3$ is a 3×1 column matrix. These are ordered triples. For example

$$\mathbf{a} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \quad \text{or} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

A vector in $\mathbb{R}^n$ for $n \geq 2$ is a $n \times 1$ column matrix. These are ordered n-tuples. For example

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. = (x_1, x_2, x_3, \ldots, x_n)$$

The Zero Vector: is the vector whose entries are all zeros. It will be denoted by $\mathbf{0}$ or $\vec{0}$ and is not to be confused with the scalar 0.

Algebraic Properties on $\mathbb{R}^n$

For every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^n$ and scalars c and d[2]

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(iv) $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$

(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(vii) $c(d\mathbf{u})=d(c\mathbf{u})=(cd)\mathbf{u}$

(viii) $1\mathbf{u}=\mathbf{u}$

[2]The term $-\mathbf{u}$ denotes $(-1)\mathbf{u}$.

Definition: Linear Combination

A linear combination of vectors $\mathbf{v}_1, \dots \mathbf{v}_p$ in $\mathbb{R}^n$ is a vector $\mathbf{y}$ of the form

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

where the scalars $c_1, \dots, c_p$ are often called weights.

For example, suppose we have two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Some linear combinations include

$$3\mathbf{v}_1, \quad -2\mathbf{v}_1 + 4\mathbf{v}_2, \quad \frac{1}{3}\mathbf{v}_2 + \sqrt{2}\mathbf{v}_1, \quad \text{and} \quad \mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2.$$

Example

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$. Determine if $\mathbf{b}$ can be written as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

Restated, do there exist scalars c_1, c_2 such that $c_1\vec{a}_1 + c_2\vec{a}_2 = \vec{b}$? Working with it,

$$c_1 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ -2c_1 \\ -c_1 \end{bmatrix} + \begin{bmatrix} 3c_2 \\ 0 \\ 2c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 + 3c_2 \\ -2c_1 \\ -c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$$

By vector equality, this requires

$$\begin{aligned} c_1 + 3c_2 &= -2 \\ -2c_1 &= -2 \\ -c_1 + 2c_2 &= -3 \end{aligned}$$

Our question reduces to whether this linear system is consistent.

Note this has augmented matrix

$$\begin{bmatrix} 1 & 3 & -2 \\ -2 & 0 & -2 \\ -1 & 2 & -3 \end{bmatrix}.$$

This matrix has rref

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

we see that the system is consistent; in fact there is one solution $c_1 = 1$, $c_2 = -1$.

So yes, $\vec{b}$ is a linear combination of $\vec{a}_1$, $\vec{a}_2$ with weights $c_1 = 1$, $c_2 = -1$.

Some Convenient Notation

Letting $\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}$, and in general $\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$, for $j = 1, ..., n$, we can denote the $m \times n$ matrix whose columns are these vectors by

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Note that each vector $\mathbf{a}_j$ is a vector in $\mathbb{R}^m$.

Vector and Matrix Equations

The vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}. \tag{1}$$

In particular, $\mathbf{b}$ is a linear combination of the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if the linear system whose augmented matrix is given in (1) is consistent.

Definition of **Span**

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set of vectors in $\mathbb{R}^n$. The set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = \text{Span}(S).$$

It is called the **subset of $\mathbb{R}^n$ spanned by (a.k.a. generated by)** the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

To say that a vector **b** is in $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ means that there exists a set of scalars $c_1, \ldots, c_p$ such that **b** can be written as

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p.$$

Span: Equivalent Statements

If $\mathbf{b}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, then $\mathbf{b} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$. From the previous result, we know this is equivalent to saying that the vector equation

$$x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$$

has a solution. This is in turn the same thing as saying the linear system with augmented matrix

$$[\mathbf{v}_1 \ \cdots \ \mathbf{v}_p \ \mathbf{b}]$$

is consistent.

Examples

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, and $\mathbf{a}_2 = \begin{bmatrix} -1 \\ 4 \\ -2 \end{bmatrix}$.

(a) Determine if $\mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ is in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$.

This is equivalent to asking if the system w/ augmented matrix $[\vec{a}_1\ \vec{a}_2\ \vec{b}]$ is consistent.

$$[\vec{a}_1\ \vec{a}_2\ \vec{b}] = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & 2 \\ 2 & -2 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Right most column is a pivot column;

the system is inconsistent

$\vec{b}$ is not in Span$\{\vec{a}_1, \vec{a}_2\}$.

(b) Determine if $\mathbf{b} = \begin{bmatrix} 5 \\ -5 \\ 10 \end{bmatrix}$ is in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$.

$$\begin{bmatrix} 1 & -1 & 5 \\ 1 & 4 & -5 \\ 2 & -2 & 10 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

consistent

$\vec{b}$ is in span$\{\vec{a}_1, \vec{a}_2\}$. In fact $\vec{b} = 3\vec{a}_1 - 2\vec{a}_2$.