January 24 MATH 1112 sec. 54 Spring 2020

Average Rate of Change & Difference Quotients

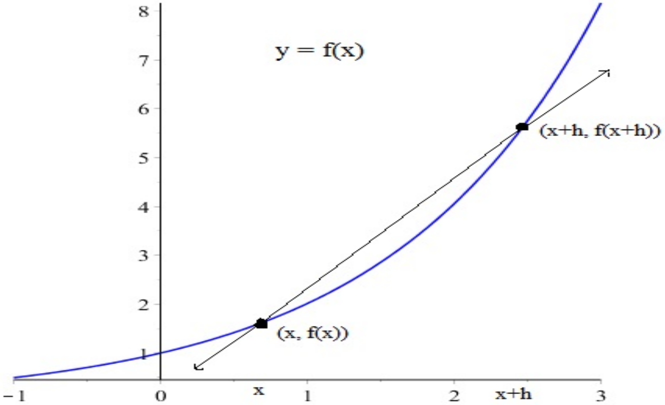

Figure: Consider points $(x, f(x))$ and $(x+h, f(x+h))$ on the graph of f and the straight line through them. This line is called a **secant** line.

Average Rate of Change: Difference Quotients

Let f be defined at x and $x+h$ for nonzero number h. The line passing through $(x, f(x))$ and $(x+h, f(x+h))$ is called a **secant** line. Determine its slope.

$(x, f(x))$ $\quad (x+h, f(x+h))$
$x_1, y_1 \quad x_2, y_2$

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{x+h-x}$$

$$= \frac{f(x+h) - f(x)}{h}$$

The result is called a **difference quotient**. It is the **average rate of change** of f on the interval $[x, x+h]$.

Example

For $h \neq 0$, construct and simplify the difference quotient $\frac{f(x+h)-f(x)}{h}$ for $f(x) = \frac{1}{x}$.

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

Clear fractions
multiply
and
divide by
$x(x+h)$

$$= \left(\frac{\frac{1}{x+h} - \frac{1}{x}}{h}\right) \frac{x(x+h)}{x(x+h)}$$

$$= \frac{\frac{1}{x+h}\left(x(x+h)\right) - \frac{1}{x} \times (x+h)}{h\,x\,(x+h)}$$

$$= \frac{x - (x+h)}{hx(x+h)}$$

$$= \frac{x - x - h}{hx(x+h)}$$

$$= \frac{-h}{hx(x+h)}$$

$$= \frac{-1}{x(x+h)}$$

Question

The difference quotient $\frac{f(x+h)-f(x)}{h}$ for $f(x) = 3x^2 - x$ is

(a) $6x + 3h - 1$

$f(x+h) = 3(x+h)^2 - (x+h)$

$= 3(x^2 + 2xh + h^2) - x - h$

$\frac{f(x+h)-f(x)}{h} = \frac{3x^2 + 6xh + 3h^2 - x - h - 3x^2 + x}{h}$

(b) $3h + 1$

(c) $\frac{3h^2 - 2x + h}{h}$

(d) $6x - 1$

(e) none of the above is the correct answer

Compositions

Suppose a spherical balloon is inflated so that the radius after time t seconds is given by the function $r(t) = 2t$ cm. The volume of a sphere of radius r is known to be $V(r) = \frac{4}{3}\pi r^3$. Note that

- r is a function of t, and
- V is a function of r, making
- V a function of t (through its dependence on r). In fact,

$$V(t) = V(r(t)) = \frac{4}{3}\pi(2t)^3 = \frac{32}{3}\pi t^3.$$

This is an example of a **composition** of functions.

Composition: Definition and Notation

Let f and g be functions. Then the **composite** function denoted

$$f \circ g,$$

also called the **composition** of f and g, is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

The expression $f \circ g$ is read "f composed with g", and $(f \circ g)(x)$ is read "f of g of x".

Example

Let $f(x) = \sqrt{x-1}$ and $g(x) = \frac{2}{x+1}$. Evaluate each expression if possible.

$g(1) = \frac{2}{1+1} = 1$

(a) $(f \circ g)(1)$ $= f(g(1)) = f(1) = \sqrt{1-1} = 0$

(b) $(g \circ f)(1)$ $= g(f(1)) = g(0) = \frac{2}{0+1} = 2$

$f(1) = \sqrt{1-1} = 0$

(c) $(f \circ g)(0)$ $= f(g(0)) = f(2) = \sqrt{2-1} = 1$

(d) $(g \circ f)(0)$ $= g(f(0))$ is undefined since $f(0)$ is undefined.

$f(0) = \sqrt{0-1}$ undefined

Question

True/False: Suppose f and g are functions. The compositions

$$f \circ g \quad \text{and} \quad g \circ f$$

are equal.

(a) True, and I'm confident

(b) True, but I'm not sure

false!

(c) False, and I'm confident

(d) False, but I'm not sure