January 24 Math 3260 sec. 51 Spring 2020

Section 1.4: The Matrix Equation $A\mathbf{x} = \mathbf{b}$.

Definition Let A be an $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ (each in $\mathbb{R}^m$), and let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. Then the product of A and $\mathbf{x}$, denoted by

$$A\mathbf{x}$$

$A_{m\times n}\, x_{n\times 1} \to m\times 1$

is the linear combination of the columns of A whose weights are the corresponding entries in $\mathbf{x}$. That is

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

(Note that the result is a vector in $\mathbb{R}^m$!)

Example

Find the product $A\mathbf{x}$. Simplify to the extent possible.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ -2 & -1 & 4 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$A\vec{x} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3$$

$$= 2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + (-1) \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0+3 \\ -4-1-4 \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \end{bmatrix} \quad \text{in } \mathbb{R}^2$$

Example

Find the product $A\mathbf{x}$. Simplify to the extent possible.

$$A = \begin{bmatrix} 2 & 4 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2$$

$$= -3\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 8 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

in $\mathbb{R}^3$

Example

Write the linear system as a vector equation and then as a matrix equation of the form $A\mathbf{x} = \mathbf{b}$.

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & x_3 & = & 2 \\ x_1 & + & x_2 & + & & = & -1 \end{array}$$

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

coefficient matrix for the system

$$\underbrace{\begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{\vec{b}}$$

Form $A\vec{x} = \vec{b}$

Theorem

If A is the $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$, and $\mathbf{b}$ is in $\mathbb{R}^m$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

$\underset{\sim}{x}$ is a variable vector in $\mathbb{R}^n$.

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the linear system of equations whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}].$$

Corollary

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

In other words, the corresponding linear system is consistent if and only if $\mathbf{b}$ is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

Example

Characterize the set of all vectors $\mathbf{b} = (b_1, b_2, b_3)$ such that $A\mathbf{x} = \mathbf{b}$ has a solution where

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}.$$

The eqn $A\vec{x} = \vec{b}$ is equivalent to the linear system w/ augmented matrix $[A\ \vec{b}]$.

$$[A\ \vec{b}] = \begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix}$$

Let's do row reduction to an ref.

$4R_1 + R_2 \to R_2$

$3R_1 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2+4b_1 \\ 0 & 7 & 5 & b_3+3b_1 \end{bmatrix}$$

$$-2R_3 \to R_3$$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2+4b_1 \\ 0 & -14 & -10 & -2b_3-6b_1 \end{bmatrix}$$

$$R_2 + R_3 \to R_3$$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2+4b_1 \\ 0 & 0 & 0 & b_2+4b_1-2b_3-6b_1 \end{bmatrix}$$

The system is consistent provided the 4th column is <u>not</u> a pivot column. This requires

$$-2b_1 + b_2 - 2b_3 = 0$$

The system $A\vec{x} = \vec{b}$ is consistent provided

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{where} \quad -2b_1 + b_2 - 2b_3 = 0$$

This is a linear system with augmented matrix

$$\begin{bmatrix} -2 & 1 & -2 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -\frac{1}{2} & 1 & 0 \end{bmatrix}$$

$$b_1 = \frac{1}{2} b_2 - b_3$$

b_2, b_3 – free

So $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}b_2 - b_3 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}b_2 \\ b_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -b_3 \\ 0 \\ b_3 \end{bmatrix}$

$$= b_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$A\vec{x} = \vec{b}$ has a solution if and only if

$\vec{b}$ is in $\text{Span}\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$