January 24 Math 3260 sec. 55 Spring 2020

Section 1.4: The Matrix Equation $A\mathbf{x} = \mathbf{b}$.

Definition Let A be an $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ (each in $\mathbb{R}^m$), and let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. Then the product of A and $\mathbf{x}$, denoted by

$$A\mathbf{x}$$

is the linear combination of the columns of A whose weights are the corresponding entries in $\mathbf{x}$. That is

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

(Note that the result is a vector in $\mathbb{R}^m$!)

Example

Find the product $A\mathbf{x}$. Simplify to the extent possible.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ -2 & -1 & 4 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

2×3 in $\mathbb{R}^3$

$A\vec{x}$

$2\times 3 \quad 3\times 1 \to 2\times 1$

$$A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3$$

$$= 2\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 1\begin{bmatrix} 0 \\ -1 \end{bmatrix} + (-1)\begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0+3 \\ -4-1-4 \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \end{bmatrix}$$

in $\mathbb{R}^2$

Example

Find the product $A\mathbf{x}$. Simplify to the extent possible.

$$A = \begin{bmatrix} 2 & 4 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

in $\mathbb{R}^2$

3×2

$$A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 = -3\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -6 + 8 \\ 3 + 2 \\ 0 + 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

Example

Write the linear system as a vector equation and then as a matrix equation of the form $A\mathbf{x} = \mathbf{b}$.

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & x_3 & = & 2 \\ x_1 & + & x_2 & + & & = & -1 \end{array}$$

writing as a vector equation

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

equivalent to

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

A is the coefficient matrix for the system

$$A \vec{x} = \vec{b}$$

Theorem

If A is the $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$, and $\mathbf{b}$ is in $\mathbb{R}^m$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

$\vec{x}$ is a variable vector

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the linear system of equations whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}].$$

Corollary

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

In other words, the corresponding linear system is consistent if and only if $\mathbf{b}$ is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$.

Example

Characterize the set of all vectors $\mathbf{b} = (b_1, b_2, b_3)$ such that $A\mathbf{x} = \mathbf{b}$ has a solution where

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}.$$

We can use the augmented matrix $[A\ \vec{b}]$ to investigate solvability.

$$[A\ \vec{b}] = \begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix}$$

Look for an ref

$4R_1 + R_2 \to R_2$

$3R_1 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2+4b_1 \\ 0 & 7 & 5 & b_3+3b_1 \end{bmatrix} \qquad -2R_3 \to R_3$$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2+4b_1 \\ 0 & -14 & -10 & -2b_3-6b_1 \end{bmatrix}$$

$$R_2 + R_3 \to R_3$$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2+4b_1 \\ 0 & 0 & 0 & -2b_3-6b_1+b_2+4b_1 \end{bmatrix}$$

The system is consistent provided the 4th column is not a pivot column.

That holds if $-2b_3 - 2b_1 + b_2 = 0$

$A\vec{x} = \vec{b}$ is consistent if $\vec{b}$ solves the system $-2b_1 + b_2 - 2b_3 = 0$

This has augmented matrix

$$\begin{bmatrix} -2 & 1 & -2 & 0 \end{bmatrix} \xrightarrow[-\frac{1}{2}R_1 \to R_1]{\text{rref}} \begin{bmatrix} 1 & -\frac{1}{2} & 1 & 0 \end{bmatrix}$$

b_1 is basic

b_2, b_3 are free

$b_1 = \frac{1}{2}b_2 - b_3$

b_2, b_3 - free

A solution

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}b_2 - b_3 \\ b_2 \\ b_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}b_2 \\ b_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -b_3 \\ 0 \\ b_3 \end{bmatrix}$$

$$= b_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

We can express the set of such $\vec{b}$ as

$$\text{Span}\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$