January 25 Math 2306 sec. 53 Spring 2019

Section 4: Bernoulli Equations

Suppose $P(x)$ and $f(x)$ are continuous on some interval (a, b) and n is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is called a **Bernoulli** equation.

Observation: This equation has the flavor of a linear ODE, but since $n \neq 0, 1$ it is necessarily nonlinear. So our previous approach involving an integrating factor does not apply directly. Fortunately, we can use a change of variables to obtain a related linear equation.

Solving the Bernoulli Equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

Let the new dependent variable u be defined by

$u = y^{1-n}$. Then $\frac{du}{dx} = (1-n)y^{1-n-1}\frac{dy}{dx} = (1-n)y^{-n}\frac{dy}{dx}$

Divide the ODE by y^n

$$y^{-n}\frac{dy}{dx} + P(x)y^{-n}y = f(x)y^n y^{-n}$$

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = f(x)$$

The equation for u is

$$\frac{1}{1-n}\frac{du}{dx} + P(x)\,u = f(x)$$

$*$ $n \neq 1$

so $1-n \neq 0$

u solves the 1st order linear equation

$$\frac{du}{dx} + (1-n)P(x)\,u = (1-n)f(x)$$

The form here is

$$\frac{du}{dx} + P_1(x)\,u = f_1(x)$$

where $P_1(x) = (1-n)P(x)$ and $f_1(x) = (1-n)f(x)$

Solve this using an integrating factor.

Since $u = y^{1-n}$, $y = u^{\frac{1}{1-n}}$.

Example

Solve the initial value problem $y' - y = -e^{2x}y^3$, subject to $y(0) = 1$.

The Bernoulli equation $\frac{dy}{dx} - y = -e^{2x}y^3$ has $n=3$.

Let $u = y^{1-3} = y^{-2}$. Then $\frac{du}{dx} = -2y^{-3}\frac{dy}{dx}$.

Dividing by y^3

$$\underbrace{y^{-3}\frac{dy}{dx}}_{-\frac{1}{2}\frac{du}{dx}} - \underbrace{y^{-3}y}_{u} = -e^{2x}\underbrace{y^3y^{-3}}_{1}$$

u's equation is

$$\frac{-1}{2}\frac{du}{dx} - u = -e^{2x}$$

In standard form $\quad \frac{du}{dx} + 2u = 2e^{2x} \qquad P_1(x) = 2$

The integrating factor $\quad \mu = e^{\int P_1(x)\,dx} = e^{\int 2\,dx} = e^{2x}$

Multiply by μ

$$\frac{d}{dx}\left(e^{2x}u\right) = 2e^{2x}e^{2x} = 2e^{4x}$$

$$\int \frac{d}{dx}\left(e^{2x}u\right)dx = \int 2e^{4x}\,dx$$

$$e^{2x}u = \frac{1}{2}e^{4x} + C$$

$$u = \frac{\frac{1}{2}e^{4x} + C}{e^{2x}} = \frac{1}{2}e^{2x} + Ce^{-2x}$$

Since $u = y^{-2}$, $y = u^{-1/2} = \frac{1}{\sqrt{u}}$. Hence

$$y = \frac{1}{\sqrt{\frac{1}{2}e^{2x} + Ce^{-2x}}}.$$ Now apply $y(0) = 1$.

$$1 = \frac{1}{\sqrt{\frac{1}{2}e^{0} + Ce^{0}}} = \frac{1}{\sqrt{\frac{1}{2} + C}}$$

$$\sqrt{\frac{1}{2} + C} = 1$$

$$\frac{1}{2} + C = 1^2 = 1 \implies C = 1 - \frac{1}{2} = \frac{1}{2}$$

The solution to the IVP is

$$y = \frac{1}{\sqrt{\frac{1}{2}e^{2x}+\frac{1}{2}e^{-2x}}} = \frac{1}{\sqrt{\frac{1}{2}}\sqrt{e^{2x}+e^{-2x}}}$$

$$\boxed{y = \frac{\sqrt{2}}{\sqrt{e^{2x}+e^{-2x}}}}$$

Exact Equations

We considered first order equations of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0. \tag{1}$$

The left side is called a *differential form*. We will assume here that M and N are continuous on some (shared) region in the plane.

Definition: The equation (1) is called an **exact equation** on some rectangle R if there exists a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x,y)$$

Derivative of F with respect to x holding y constant

Derivative of F wrt y holding x constant

for every (x,y) in R.

Exact Equation Solution

If $M(x,y)\,dx + N(x,y)\,dy = 0$ happens to be exact, then it is equivalent to

$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 0$$

This implies that the function F is constant on R and solutions to the DE are given by the relation

$$F(x,y) = C$$

This equation defines solutions implicitly.

Recognizing Exactness

There is a theorem from calculus that ensures that if a function F has first partials on a domain, and if those partials are continuous, then the second mixed partials are equal. That is,

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}.$$

If $\frac{\partial F}{\partial x} = M$ then

$$\frac{\partial^2 F}{\partial y\, \partial x} = \frac{\partial M}{\partial y}$$

If it is true that

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N$$

this provides a condition for exactness, namely

and

If $\frac{\partial F}{\partial y} = N$, then

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial^2 F}{\partial x\, \partial y} = \frac{\partial N}{\partial x}$$

Exact Equations

Theorem: Let M and N be continuous on some rectangle R in the plane. Then the equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Example

Show that the equation is exact and obtain a family of solutions.

$$(2xy - \sec^2 x)\,dx + (x^2 + 2y)\,dy = 0$$

$M(x,y) = 2xy - \sec^2 x$ and $N(x,y) = x^2 + 2y$

$$\frac{\partial M}{\partial y} = 2x \cdot 1 - 0 = 2x$$

Treat x as constant

$$\frac{\partial N}{\partial x} = 2x + 0 = 2x$$

Treat y as constant

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$ the equation is exact.

Solutions to the ODE will have the form

$F(x,y) = C$ for some function F.

We know that $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$.

$$\frac{\partial F}{\partial x} = 2xy - \sec^2 x$$

$$\frac{\partial F}{\partial y} = x^2 + 2y$$

This is enough information to find $F(x,y)$

up to added constant.