January 25 Math 2306 sec. 57 Spring 2018

Section 4: First Order Equations: Linear

We will find the general solution of the first order linear equation in **standard form**

$$\frac{dy}{dx} + P(x)y = f(x).$$

We assume that P and f are continuous on the domain of definition. The general solution (a 1-parameter family) will have the form

$$y = y_c + y_p$$

where y_c is the complementary solution to the associated homogeneous equation ($y' + Py = 0$) and y_p is called a particular solution.

Derivation of Solution via Integrating Factor

Solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

We will multiply by a certain function $\mu(x)$ such that the resulting left side will be a single derivative, product rule term. It will be

$$\frac{d}{dx}\left[\mu(x)\, y\right].$$

Mult. by μ

$$\underbrace{\mu \frac{dy}{dx} + \mu P(x) y}_{\text{want this to be } \frac{d}{dx}[\mu y].} = \mu f(x)$$

$$\frac{d}{dx}\left[\mu y\right] = \mu \frac{dy}{dx} + \frac{d\mu}{dx} y$$

Equality of the 2^{nd} terms requires

$$\frac{d\mu}{dx} y = \mu P(x) y$$

Cancel the y's, and assume $\mu(x) > 0$.

$$\frac{d\mu}{dx} = \mu P(x)$$

1^{st} order separable!

Separate Variables

$$\frac{1}{\mu} d\mu = P(x)\, dx$$

$$\int \frac{1}{\mu}\, d\mu = \int P(x)\,dx \Rightarrow \ln\mu = \int P(x)\,dx$$

$$\boxed{\mu = e^{\int P(x)\,dx} \qquad \mu \text{ is called an integrating factor}}$$

Now we solve the ODE. Mult. by μ

$$\mu \frac{dy}{dx} + \mu P(x)\, y = \mu f(x)$$

guaranteed by choice of μ

$$\frac{d}{dx}[\mu y] = \mu f(x)$$

Integrate $$\int \frac{d}{dx}\left[\mu y\right] dx = \int \mu(x) f(x)\, dx$$

$$\mu y = \int \mu(x) f(x)\, dx + C$$

Isolate y

$$\boxed{y = \underbrace{\frac{1}{\mu}\int \mu(x) f(x)\, dx}_{y_p} + \underbrace{\frac{C}{\mu}}_{y_c}}$$

In terms of the exponential w/ P

$$y = e^{-\int P(x)\,dx} \int e^{\int P(x)\,dx} f(x)\,dx + C e^{-\int P(x)\,dx}$$

General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx} f(x)\,dx + C\right)$$

Solve the ODE

$$\frac{dy}{dx} + y = 3xe^{-x}$$

It's in standard form. $P(x) = 1$

Get $\mu = e^{\int P(x)\,dx} = e^{\int 1\,dx} = e^{x+C} = e^{x} \cdot e^{C} = ke^{x}$

where $k = e^{C}$. (we'll see that we can take $C=0$, $k=1$.)

$\mu = ke^{x}$. Mult. by μ

$$ke^{x}\frac{dy}{dx} + ke^{x}y = 3xe^{-x}(ke^{x}) = 3kx \qquad e^{-x}\cdot e^{x} = 1$$

Cancel k: $\quad e^{x}\frac{dy}{dx} + e^{x}y = 3x$

$$\frac{d}{dx}\left[e^x y\right] = 3x$$

Integrate

$$\int \frac{d}{dx}\left[e^x y\right]dx = \int 3x\,dx$$

$$e^x y = \frac{3}{2}x^2 + C$$

$$y = \frac{\frac{3}{2}x^2 + C}{e^x} = \frac{3}{2}x^2e^{-x} + Ce^{-x}$$

Our solution is

$$\boxed{y = \frac{3}{2}x^2e^{-x} + Ce^{-x}}$$

Solve the IVP

$$x\frac{dy}{dx} - y = 2x^2, \quad x > 0 \quad y(1) = 5$$

Put in standard form:

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{2x^2}{x} = 2x \quad \Rightarrow \quad \frac{dy}{dx} + \left(\frac{-1}{x}\right)y = 2x$$

$$P(x) = \frac{-1}{x}. \qquad \mu = e^{\int P(x)\,dx} = e^{\int \frac{-1}{x}dx} = e^{-\int \frac{1}{x}dx} = e^{-\ln x}$$

$$= e^{\ln x^{-1}} = x^{-1} \qquad \boxed{\mu = x^{-1}}$$

$$x^{-1}\frac{dy}{dx} - x^{-1}\cdot\frac{1}{x}y = x^{-1}(2x)$$

$$\frac{d}{dx}\left[x^{-1}y\right] = 2$$

$$\int \frac{d}{dx}\left[x^{-1}y\right]dx = \int 2\,dx$$

$$x^{-1}y = 2x + C$$

$$y = \frac{2x+C}{x^{-1}} = 2x^2 + Cx$$

The 1-parameter family of solutions to the ODE is

$$y = 2x^2 + Cx.$$

To solve the IVP, we apply $y(1) = 5$.

$$y(1) = 2(1)^2 + C(1) = 5 \Rightarrow 2 + C = 5$$

$$C = 3$$

The solution to the IVP is

$$\boxed{y = 2x^2 + 3x.}$$

Verify

Just for giggles, lets verify that our solution $y = 2x^2 + 3x$ really does solve the differential equation we started with

$$x\frac{dy}{dx} - y = 2x^2.$$

If $y = 2x^2 + 3x$, then $\frac{dy}{dx} = 4x + 3$

So $xy' - y = x(4x+3) - (2x^2 + 3x)$

$= 4x^2 + 3x - 2x^2 - 3x$

$= 4x^2 - 2x^2 + \cancel{3}x - \cancel{3}x$

$= 2x^2$

Steady and Transient States

For some linear equations, the term y_c decays as x (or t) grows. For example

$$\frac{dy}{dx} + y = 3xe^{-x} \quad \text{has solution} \quad y = \frac{3}{2}x^2e^{-x} + Ce^{-x}.$$

$$\text{Here,} \quad y_p = \frac{3}{2}x^2e^{-x} \quad \text{and} \quad y_c = Ce^{-x}.$$

Such a decaying complementary solution is called a **transient state.**

The corresponding particular solution is called a **steady state.**