Section 4: First Order Equations: Linear

We will find the general solution of the first order linear equation in standard form

\[ \frac{dy}{dx} + P(x)y = f(x). \]

We assume that \( P \) and \( f \) are continuous on the domain of definition. The general solution (a 1-parameter family) will have the form

\[ y = y_c + y_p \]

where \( y_c \) is the complementary solution to the associated homogeneous equation \( (y' + Py = 0) \) and \( y_p \) is called a particular solution.
Derivation of Solution via Integrating Factor

Solve the equation in standard form

\[ \frac{dy}{dx} + P(x)y = f(x) \]

we will multiply by a certain function \( \mu(x) \) such that the resulting left side will be a single derivative, product rule term. It will be

\[ \frac{d}{dx} \left[ \mu(x) y \right] . \]

Mult. by \( \mu \)

\[ \mu \frac{dy}{dx} + \mu P(x)y = \mu f(x) \]

\[ \text{want this to be } \frac{d}{dx} \left[ \mu y \right] . \]
\[
\frac{d}{dx} \left[ \mu y \right] = \mu \frac{dy}{dx} + \frac{d\mu}{dx} y
\]

Equality of the 2nd terms requires

\[
\frac{d\mu}{dx} y = \mu P(x) y
\]

Cancel the \(y\)'s, and assume \(\mu(x) > 0\).

\[
\frac{d\mu}{dx} = \mu P(x)
\]

1st order separable!

Separate Variables

\[
\frac{1}{\mu} d\mu = P(x) dx
\]
\[ \int \frac{1}{\mu} \, d\mu = \int P(x) \, dx \Rightarrow \ln \mu = \int P(x) \, dx \]

\[ \int P(x) \, dx \]

\[ \mu = e \]

\[ \mu \text{ is called an integrating factor} \]

Now we solve the ODE. multi. by \( \mu \)

\[ \mu \frac{dy}{dx} + \mu P(x) y = \mu f(x) \]

\[ \frac{d}{dx} \left[ \mu y \right] = \mu f(x) \]
Integrate

\[ \int \frac{d}{dx} [\mu y] \, dx = \int \mu(x) f(x) \, dx \]

\[ \mu y = \int \mu(x) f(x) \, dx + C \]

Isolate \( y \)

\[ y = \frac{1}{\mu} \int \mu(x) f(x) \, dx + \frac{C}{\mu} \]

\( y_p \)

\( y_c \)
In terms of the exponential $e^P$

\[ y = e^{-\int p(x) \, dx} \int e^{\int p(x) \, dx} f(x) \, dx + C e^{-\int p(x) \, dx} \]
General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor $\mu$. The left hand side will always collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for $y$.

$$y(x) = \frac{1}{\mu(x)} \int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx} \left(\int e^{\int P(x)\,dx}f(x)\,dx + C\right)$$
Solve the ODE

\[
\frac{dy}{dx} + y = 3xe^{-x}
\]

It's in standard form. \( P(x) = 1 \)

Get \( \mu = e^{\int P(x) \, dx} = e^{\int 1 \, dx} = e^x = e \cdot e = ke \)

where \( k = e^c \). (we'll see that we can take \( c = 0, k = 1 \)).

\[
\mu = ke^x. \text{ Multi. by } \mu
\]

\[
ke^x \frac{dy}{dx} + ke^x y = 3xe^{-x} (ke^x) = 3ke^x \quad e^x \cdot e = 1
\]

Cancel \( k \):

\[
\frac{dy}{dx} + e^x y = 3x
\]
\[ \frac{d}{dx} \left[ e^y \right] = 3x \]

Integrate:
\[ \int \frac{d}{dx} \left[ e^y \right] \, dx = \int 3x \, dx \]

\[ e^y = \frac{3}{2} x^2 + C \]

\[ y = \frac{\frac{3}{2} x^2 + C}{e^x} = \frac{3}{2} x^2 e^{-x} + Ce^{-x} \]

Our solution is:
\[ y = \frac{3}{2} x^2 e^{-x} + Ce^{-x} \]
Solve the IVP

\[ x \frac{dy}{dx} - y = 2x^2, \quad x > 0 \quad y(1) = 5 \]

Put in standard form:

\[ \frac{dy}{dx} - \frac{1}{x} y = \frac{2x^2}{x} = 2x \quad \Rightarrow \quad \frac{dy}{dx} + \left( \frac{-1}{x} \right) y = 2x \]

\[ P(x) = \frac{-1}{x}. \quad \mu = e^{-\int P(x) \, dx} = e^{-\int \frac{-1}{x} \, dx} = e^{\ln x^{-1}} = x^{-1} \]

\[ x^{-1} \frac{dy}{dx} - x^{-1} \cdot \frac{1}{x} y = x^{-1} (2x) \]

\[ \frac{d}{dx} \left[ x^{-1} y \right] = 2 \]
\[
\int \frac{1}{x} \ln(x) \, dx = \int 2 \, dx
\]

\[
\ln x = 2x + C
\]

\[
y = 2x^2 + Cx
\]

The 1-parameter family of solutions to the ODE is

\[
y = 2x^2 + Cx,
\]

To solve the IVP, we apply \( y(1) = 5 \).
\[ y(1) = 2(1)^2 + C(1) = 5 \implies 2 + C = 5 \]
\[ C = 3 \]

The solution to the IVP is

\[ y = 2x^2 + 3x \]
Verify

Just for giggles, let's verify that our solution \( y = 2x^2 + 3x \) really does solve the differential equation we started with

\[
x \frac{dy}{dx} - y = 2x^2.
\]

If \( y = 2x^2 + 3x \), then \( \frac{dy}{dx} = 4x + 3 \)

So,

\[
x y' - y = x(4x + 3) - (2x^2 + 3x)
\]

\[
= 4x^2 + 3x - 2x^2 - 3x
\]

\[
= 4x^2 - 2x^2 + 3x - 3x
\]

\[
= 2x^2
\]
For some linear equations, the term $y_c$ decays as $x$ (or $t$) grows. For example

$$\frac{dy}{dx} + y = 3xe^{-x} \quad \text{has solution} \quad y = \frac{3}{2} x^2 e^{-x} + Ce^{-x}.$$

Here, $y_p = \frac{3}{2} x^2 e^{-x}$ and $y_c = Ce^{-x}$.

Such a decaying complementary solution is called a transient state.

The corresponding particular solution is called a steady state.