January 25 Math 2306 sec. 60 Spring 2018

Section 4: First Order Equations: Linear

We will find the general solution of the first order linear equation in **standard form**

$$\frac{dy}{dx} + P(x)y = f(x).$$

We assume that P and f are continuous on the domain of definition. The general solution (a 1-parameter family) will have the form

$$y = y_c + y_p$$

where y_c is the complementary solution to the associated homogeneous equation ($y' + Py = 0$) and y_p is called a particular solution.

Derivation of Solution via Integrating Factor

Solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

We multiply the equation by a function $\mu(x)$ in such a way that the left side becomes one derivative term, the product rule $\frac{d}{dx}\left[\mu(x)y(x)\right]$. Let's assume $\mu(x) > 0$.

Mult. by μ

$$\underbrace{\mu\frac{dy}{dx} + \mu P(x)y}_{\text{want this to be}} = \mu f(x)$$

$$\frac{d}{dx}\left[\mu y\right].$$

$$\frac{d}{dx}\left[\mu y\right] = \mu \frac{dy}{dx} + \frac{d\mu}{dx} y .$$ This matches the left side above if $$\frac{d\mu}{dx} y = \mu P(x) y$$

Canceling y: $$\frac{d\mu}{dx} = \mu P(x)$$ a separable equation for μ

Separate variables
$$\frac{1}{\mu}\frac{d\mu}{dx} = P(x)$$

$$\int \frac{1}{\mu}\, d\mu = \int P(x)\, dx$$

$$\ln \mu = \int P(x)\,dx \quad \Rightarrow \quad \boxed{\mu = e^{\int P(x)\,dx}}$$

This is called an integrating factor.

Now we can solve for y: With our choice of μ

$$\mu \frac{dy}{dx} + \mu P(x) y = \mu f(x)$$

$$\frac{d}{dx}\left[\mu y\right] = \mu f(x)$$

Integrate and divide by μ

$$\int \frac{d}{dx}\left[\mu y\right] dx = \int \mu(x) f(x)\, dx$$

$$\mu y = \int \mu(x) f(x)\, dx + C$$

$$y = \underbrace{\frac{1}{\mu} \int \mu(x) f(x)\, dx}_{y_p} + \underbrace{\frac{C}{\mu}}_{y_c}$$

We can write this as

$$y = e^{-\int P(x)\,dx} \int \left(e^{\int P(x)\,dx} f(x)\right) dx + C e^{-\int P(x)\,dx}$$

General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx}f(x)\,dx + C\right)$$

Solve the ODE

$$\frac{dy}{dx}+y=3xe^{-x}$$

It's in standard form: $P(x)=1$

Find $\mu = e^{\int P(x)\,dx} = e^{\int dx} = e^{x+c} = e^{c}\cdot e^{x} = ke^{x}$

where $k=e^{c}$. We'll see that we can take $c=0$, $k=1$,

$\mu = ke^{x}$. Mult. eqn

$$ke^{x}\frac{dy}{dx}+ke^{x}y=ke^{x}(3xe^{-x})$$

Cancel the k's $\quad e^{x}\frac{dy}{dx}+e^{x}y=3xe^{x}\cdot e^{-x}=3x$

$$\frac{d}{dx}\left[e^{x}y\right]=3x$$

$$\int \frac{d}{dx}\left[e^{x}y\right]dx = \int 3x\,dx$$

$$e^{x}y = \frac{3}{2}x^2 + C$$

$$y = \frac{\frac{3}{2}x^2 + C}{e^{x}}$$

The solutions to the ODE is given by

$$y = \underbrace{\frac{3}{2}x^2e^{-x}}_{y_p} + \underbrace{Ce^{-x}}_{y_c}$$

Solve the IVP

$$x\frac{dy}{dx}-y=2x^2,\ x>0\quad y(1)=5$$

Put in standard form $\frac{dy}{dx}-\frac{1}{x}y=\frac{2x^2}{x}=2x$

$$\frac{dy}{dx}+\left(\frac{-1}{x}\right)y=2x \qquad P(x)=\frac{-1}{x}$$

Find $\mu=e^{\int P(x)\,dx}=e^{\int \frac{-1}{x}dx}$

$$=e^{-\int \frac{1}{x}dx}=e^{-\ln x}=e^{\ln x^{-1}}=x^{-1}$$

$$\mu=x^{-1}$$

Multiply by μ:

$$x^{-1}\frac{dy}{dx} - x^{-1}\cdot\frac{1}{x}y = x^{-1}(2x)$$

$$\frac{d}{dx}\left[x^{-1}y\right] = 2$$

$$\int \frac{d}{dx}\left[x^{-1}y\right]dx = \int 2\,dx$$

$$x^{-1}y = 2x + C$$

$$y = \frac{2x+C}{x^{-1}} = 2x^2 + Cx$$

The 1-parameter family of solutions to the ODE is

$$y = 2x^2 + Cx$$

Apply $y(1) = 5$.

$$y(1) = 2(1^2) + C(1) = 5$$

$$2 + C = 5$$

$$C = 3.$$

The solution to the IVP is

$$y = 2x^2 + 3x.$$

Verify

Just for giggles, lets verify that our solution $y = 2x^2 + 3x$ really does solve the differential equation we started with

$$x\frac{dy}{dx} - y = 2x^2.$$

$y = 2x^2 + 3x, \qquad y' = 4x + 3$

$$x\frac{dy}{dx} - y = x(4x+3) - (2x^2 + 3x)$$

$$= 4x^2 + 3x - 2x^2 - 3x$$

$$= 4x^2 - 2x^2 + \cancel{3x} - \cancel{3x} = 2x^2$$

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