January 25 Math 2306 sec. 60 Spring 2019

Section 4: Bernoulli Equations

Suppose $P(x)$ and $f(x)$ are continuous on some interval (a, b) and n is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is called a **Bernoulli** equation.

Observation: This equation has the flavor of a linear ODE, but since $n \neq 0, 1$ it is necessarily nonlinear. So our previous approach involving an integrating factor does not apply directly. Fortunately, we can use a change of variables to obtain a related linear equation.

Solving the Bernoulli Equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

Define a new dependent variable u by

$u = y^{1-n}$. Note that $\frac{du}{dx} = (1-n)y^{1-n-1}\frac{dy}{dx}$

So $u = y^{1-n}$ and $\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx}$.

Divide the ODE by y^n

$$y^{-n}\frac{dy}{dx} + P(x)y^{-n}y = f(x)y^n y^{-n}$$

$$y^{-n}\frac{dy}{dx} + P(x)\, y^{1-n} = f(x)$$

$n \neq 1$ so

$1-n \neq 0$

$$\frac{1}{1-n}\frac{du}{dx} + P(x)\, u = f(x)$$

The equation is 1^{st} order linear for u

$$\frac{du}{dx} + (1-n)P(x)\,u = (1-n)\, f(x).$$

This form is

$$\frac{du}{dx} + P_1(x)\,u = f_1(x)$$

where $P_1(x) = (1-n)P(x)$ and $f_1(x) = (1-n)f(x)$

Solve this equation for u using an integrating factor. Then from

$$u = y^{1-n}, \text{ we have}$$

$$y = u^{\frac{1}{1-n}}$$

Example

Solve the initial value problem $y' - y = -e^{2x}y^3$, subject to $y(0) = 1$.

First solve the Bernoulli equation

$$\frac{dy}{dx} - y = -e^{2x}y^3 \quad \text{where} \quad n = 3.$$

Set $u = y^{1-3} = y^{-2}$. $\frac{du}{dx} = -2y^{-3}\frac{dy}{dx}$. Divide the ODE by y^3

$$\underbrace{y^{-3}\frac{dy}{dx}}_{-\frac{1}{2}\frac{du}{dx}} - \underbrace{y^{-3}y}_{y^{-2}} = -e^{2x}y^3y^{-3}$$

$$-\frac{1}{2}\frac{du}{dx} - u = -e^{2x} \Rightarrow \frac{du}{dx} + 2u = 2e^{2x}$$

$$P_1(x) = 2, \quad \mu = e^{\int P_1(x)\,dx} = e^{\int 2\,dx} = e^{2x}$$

Using the integrating factor

$$\frac{d}{dx}\left(e^{2x}u\right) = 2e^{2x}e^{2x} = 2e^{4x}$$

$$\int \frac{d}{dx}\left(e^{2x}u\right)dx = \int 2e^{4x}\,dx$$

$$e^{2x}u = \frac{1}{2}e^{4x} + C$$

$$u = \frac{\frac{1}{2}e^{4x} + C}{e^{2x}} = \frac{1}{2}e^{2x} + Ce^{-2x}$$

From $u = y^{-2}$, $y = u^{-1/2} = \frac{1}{\sqrt{u}}$ so

$$y = \frac{1}{\sqrt{\frac{1}{2}e^{2x} + Ce^{-2x}}}.\quad \text{Apply } y(0) = 1$$

$$1 = \frac{1}{\sqrt{\frac{1}{2}e^{0} + Ce^{0}}} = \frac{1}{\sqrt{\frac{1}{2} + C}}$$

$$\sqrt{\frac{1}{2} + C} = 1$$

$$\frac{1}{2} + C = 1^2 = 1$$

$$C = 1 - \frac{1}{2} = \frac{1}{2}$$

$$y = \frac{1}{\sqrt{\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}}} = \frac{1}{\sqrt{\frac{1}{2}}\sqrt{e^{2x} + e^{-2x}}}$$

$$= \frac{\sqrt{2}}{\sqrt{e^{2x} + e^{-2x}}}$$

The solution to the IVP is

$$\boxed{y = \frac{\sqrt{2}}{\sqrt{e^{2x} + e^{-2x}}}}$$

Exact Equations

We considered first order equations of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0. \tag{1}$$

The left side is called a *differential form*. We will assume here that M and N are continuous on some (shared) region in the plane.

Definition: The equation (1) is called an **exact equation** on some rectangle R if there exists a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x,y)$$

the derivative of F with respect to x holding y constant

Similarly, the derivative wrt y holding x constant

for every (x,y) in R.

Exact Equation Solution

If $M(x,y)\,dx + N(x,y)\,dy = 0$ happens to be exact, then it is equivalent to

$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 0$$

This implies that the function F is constant on R and solutions to the

DE are given by the relation

$$F(x,y) = C$$

This relation would define an implicit solution to the ODE.

Recognizing Exactness

There is a theorem from calculus that ensures that if a function F has first partials on a domain, and if those partials are continuous, then the second mixed partials are equal. That is,

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}.$$

If it is true that

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N \quad \Rightarrow \quad \frac{\partial^2 F}{\partial y\, \partial x} = \frac{\partial M}{\partial y}$$

and

$$\frac{\partial^2 F}{\partial x\, \partial y} = \frac{\partial N}{\partial x}$$

this provides a condition for exactness, namely

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Exact Equations

Theorem: Let M and N be continuous on some rectangle R in the plane. Then the equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Example

Show that the equation is exact and obtain a family of solutions.

$$(2xy - \sec^2 x)\,dx + (x^2 + 2y)\,dy = 0$$

$M(x,y) = 2xy - \sec^2 x$ $\qquad$ $N(x,y) = x^2 + 2y$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(2xy - \sec^2 x\right) = 2x \cdot 1 - 0 = 2x$$

Treat x as a constant

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + 2y\right) = 2x + 0 = 2x$$

Treat y as a constant

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$ the equation is exact!

Since the equation is exact, there exists a function $F(x,y)$ so that the solutions are given by $F(x,y)=C$ for constant C.

More over,

$$\frac{\partial F}{\partial x} = M(x,y) = 2xy - \sec^2 x$$

and

$$\frac{\partial F}{\partial y} = N(x,y) = x^2 + 2y$$

This is enough information to determine F

up to added constant.