January 25 Math 3260 sec. 55 Spring 2018

Section 1.3: Vector Equations

Recall that we had defined vectors as matrices consisting of a single column. We denote the collection of such vectors with n real components by $\mathbb{R}^n$. That is, a vector $\mathbf{u}$ in $\mathbb{R}^n$ has the form

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

where each u_i is a real number called a component (or entry).

Matrices Expressed in Terms of Columns

Given a collection of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ in $\mathbb{R}^m$, we can use the notation

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]$$

to denote the $m \times n$ matrix whose columns are these vectors. That is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Algebraic Operations

For vectors $\mathbf{u}$ and $\mathbf{v}$ and scalar c, we defined the two operations

Scalar Multiplication: $c\mathbf{u}$ is the vector with components cu_i.

Vector Addition: $\mathbf{u} + \mathbf{v}$ is the vector with components $u_i + v_i$.

We also define equivalence and the zero vector

Vector Equivalence: $\mathbf{u} = \mathbf{v}$ if and only if $u_i = v_i$ for each $i = 1, \ldots, n$.

Zero Vector: Denoted by $\mathbf{0}$ or $\vec{0}$, this vector has each component equal to zero.

Algebraic Properties on $\mathbb{R}^n$

For every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^n$ and scalars c and d[1]

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(iv) $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$

(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(vii) $c(d\mathbf{u})=d(c\mathbf{u})=(cd)\mathbf{u}$

(viii) $1\mathbf{u}=\mathbf{u}$

[1] The term $-\mathbf{u}$ denotes $(-1)\mathbf{u}$.

Linear Combinations and Span

Definition: Given a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$, a **linear combination** of these vectors is a vector of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

where the scalars c_i are often called *weights*.

Definition: For a given set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$, the **span** of this set, denoted

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\},$$

is the subset of $\mathbb{R}^n$ consisting of all possible linear combinations (i.e. allowing the weights to vary over all reals).

Vector and Matrix Equations

For a given set of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$, the vector equation (in variables x_i)

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\left[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}\right]. \tag{1}$$

If this system is consistent, then we can say that the vector **b** is in $\mathrm{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$.

Example

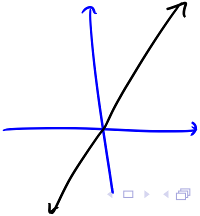
Give a geometric description of the subset of $\mathbb{R}^2$ given by $\text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Each vector in $\text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ has the form

$$\vec{x} = c \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ c \end{bmatrix}$$ for some number c.

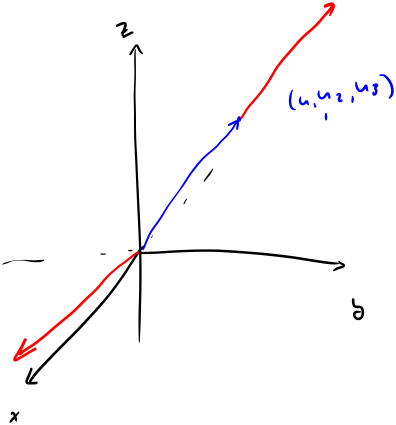
$= (c, c)$

Geometrically, this is the 45° line through the origin



Span$\{\mathbf{u}\}$ in $\mathbb{R}^3$

If $\mathbf{u}$ is any nonzero vector in $\mathbb{R}^3$, then Span$\{\mathbf{u}\}$ is a line through the origin parallel to $\mathbf{u}$.



Span$\{\mathbf{u}, \mathbf{v}\}$ in $\mathbb{R}^3$

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero, and nonparallel vectors in $\mathbb{R}^3$, then Span$\{\mathbf{u}, \mathbf{v}\}$ is a plane containing the origin parallel to both vectors.

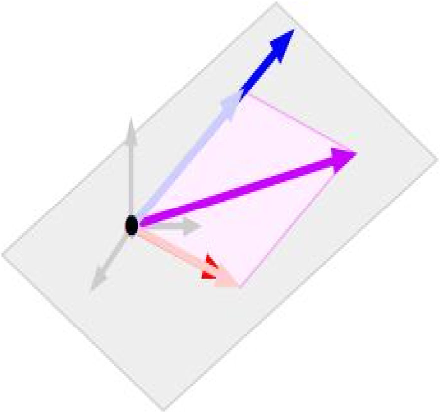


Figure: The red and blue vectors are $\mathbf{u}$ and $\mathbf{v}$. The plane is the collection of all possible linear combinations. (A purple representative is shown.)

Example

Let $\mathbf{u} = (1,1)$ and $\mathbf{v} = (0,2)$ in $\mathbb{R}^2$. Show that for every pair of real numbers a and b, that (a,b) is in Span$\{\mathbf{u},\mathbf{v}\}$.

We want to show that $x_1\vec{u} + x_2\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ is always consistent (independent of a and b).

This vector equation is the same as the linear system with augmented matrix

$$\begin{bmatrix} \vec{u} & \vec{v} & \begin{bmatrix} a \\ b \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & a \\ 1 & 2 & b \end{bmatrix}$$

Let's reduce to rref

$-R_1 + R_2 \to R_2$ $\begin{bmatrix} 1 & 0 & a \\ 0 & 2 & b-a \end{bmatrix}$

$\frac{1}{2}R_2 \to R_2$ $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & \frac{b-a}{2} \end{bmatrix}$

note, the last column is not a pivot column for any a, b pair.

Hence the system is always consistent telling us that $\begin{bmatrix} a \\ b \end{bmatrix}$ is in Span$\{\vec{u}, \vec{v}\}$ for any $\begin{bmatrix} a \\ b \end{bmatrix}$.

In fact, $\begin{bmatrix} a \\ b \end{bmatrix} = a\vec{u} + \left(\frac{b-a}{2}\right)\vec{v}$.

Section 1.4: The Matrix Equation $A\mathbf{x} = \mathbf{b}$.

Definition Let A be an $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ (each in $\mathbb{R}^m$), and let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. Then the product of A and $\mathbf{x}$, denoted by

$$A\mathbf{x}$$

is the linear combination of the columns of A whose weights are the corresponding entries in $\mathbf{x}$. That is

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

note the product requires
columns in A = # entries in $\vec{x}$

(Note that the result is a vector in $\mathbb{R}^m$!)

Example

Find the product $A\mathbf{x}$. Simplify to the extent possible.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ -2 & -1 & 4 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

3 columns 3 entries

$$A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3$$

$$= 2\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 1\begin{bmatrix} 0 \\ -1 \end{bmatrix} + (-1)\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0+3 \\ -4-1-4 \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \end{bmatrix}$$

Example

Find the product $A\mathbf{x}$. Simplify to the extent possible.

$$A = \begin{bmatrix} 2 & 4 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

2 columns

2 entries

$$A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 = -3\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 8 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

Example

Write the linear system as a vector equation and then as a matrix equation of the form $A\mathbf{x} = \mathbf{b}$.

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & x_3 & = & 2 \\ x_1 & + & x_2 & + & & = & -1 \end{array}$$

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Note that the matrix A is the coefficient matrix for the linear system. The vector $\vec{b}$ is a column for the right hand side, and

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is a variable vector.

The augmented matrix for the linear system is $\begin{bmatrix} A & \vec{b} \end{bmatrix}$.

Theorem

If A is the $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$, and $\mathbf{b}$ is in $\mathbb{R}^m$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the linear system of equations whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}].$$

Corollary

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

In other words, the corresponding linear system is consistent if and only if $\mathbf{b}$ is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

Example

Characterize the set of all vectors $\mathbf{b} = (b_1, b_2, b_3)$ such that $A\mathbf{x} = \mathbf{b}$ has a solution where

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}.$$

$A\vec{x} = \vec{b}$ has a solution if the augmented matrix $[A\ \vec{b}]$ corresponds to a consistent system.

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix}$$

We need to determine conditions under which the last column is not a pivot column.

$4R_1 + R_2 \to R_2$

$3R_1 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2+4b_1 \\ 0 & 7 & 5 & b_3+3b_1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 7 & 5 & b_3+3b_1 \\ 0 & 14 & 10 & b_2+4b_1 \end{bmatrix}$$

$-2R_2 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 7 & 5 & b_3+3b_1 \\ 0 & 0 & 0 & b_2-2b_1-2b_3 \end{bmatrix}$$

The last column is <u>not</u> a pivot column

only if

$$b_2 - 2b_1 - 2b_3 = 0$$

The vectors in $\mathbb{R}^3$ such that $A\vec{x} = \vec{b}$ is solvable satisfy

$$-2b_1 + b_2 - 2b_3 = 0.$$

Note, this can be written as

$$b_1 = \frac{1}{2} b_2 - b_3$$

b_2, b_3 free

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}b_2 - b_3 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}b_2 \\ b_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -b_3 \\ 0 \\ b_3 \end{bmatrix}$$

$$= b_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\vec{b}$ would have to be in $\text{Span}\left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Theorem (first in a string of equivalency theorems)

Let A be an $m \times n$ matrix. Then the following are logically equivalent (i.e. they are either all true or are all false).

(a) For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(b) Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

(d) A has a pivot position in every row.

(Note that statement (d) is about the *coefficient* matrix A, not about an augmented matrix $[A \quad \mathbf{b}]$.)