January 26 Math 1190 sec. 62 Spring 2017

Section 1.2: Limits of Functions Using Properties of Limits

We recall **Theorem:** If $f(x) = A$ where A is a constant, then for any real number c

$$\lim_{x\to c} f(x) = \lim_{x\to c} A = A$$

Theorem: If $f(x) = x$, then for any real number c

$$\lim_{x\to c} f(x) = \lim_{x\to c} x = c$$

Limit Law Theorems

Suppose

$$\lim_{x\to c} f(x) = L, \quad \lim_{x\to c} g(x) = M, \quad \text{and } k \text{ is constant.}$$

Theorem: (Sums) $\displaystyle\lim_{x\to c}(f(x)+g(x)) = L+M$

Theorem: (Differences) $\displaystyle\lim_{x\to c}(f(x)-g(x)) = L-M$

Theorem: (Constant Multiples) $\displaystyle\lim_{x\to c} kf(x) = kL$

Theorem: (Products) $\displaystyle\lim_{x\to c} f(x)g(x) = LM$

Theorem: (Quotient) $\displaystyle\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad \text{if } M \neq 0$

Limit Law Theorems

Theorem: (Power) $\lim_{x \to c} (f(x))^n = L^n$

Note in particular that this tells us that $\lim_{x \to c} x^n = c^n$.

Theorem: (Root) $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$ (if this is defined)

Theorem: If $R(x)$ is a rational function, and c is in the domain of R, then

$$\lim_{x \to c} R(x) = R(c).$$

Note that this includes all polynomials, and recall that the domain of any polynomial is all reals.

Observations

In limit taking, the form "$\frac{0}{0}$" sometimes appears. This is called an **indeterminate form.** Standard strategies are

(1) Try to factor the numerator and denominator to see if a common factor–$(x - c)$–can be cancelled.

(2) If dealing with roots, try rationalizing to reveal a common factor.

The form

$$\text{"}\frac{\text{nonzero constant}}{0}\text{"}$$

is not indeterminate. It is undefined. When it appears, the limit doesn't exist.

Question

$$\lim_{x\to 1} \frac{x^2 - 2x + 1}{x^2 - 1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x-1)}{\cancel{(x-1)}(x+1)}$$

$$= \lim_{x\to 1} \frac{x-1}{x+1} = \frac{0}{2} = 0$$

(a) DNE

(b) 1

(c) may exist, but can't be determined without a graph

(d) 0

Example

Let $f(x) = x^3 + 2x$. Determine the difference quotient

$$\frac{f(x+h) - f(x)}{h} \quad \text{for} \quad h \neq 0.$$

Next, take the limit as $h \to 0$ of this difference quotient.

$f(x) = x^3 + 2x$

$f(x+h) = (x+h)^3 + 2(x+h)$

$= (x+h)(x+h)^2 + 2(x+h)$

$= x^3 + 3x^2h + 3xh^2 + h^3 + 2x + 2h$

parentheses required

$$\frac{f(x+h) - f(x)}{h} = \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 2x + 2h - (x^3 + 2x)}{h}$$

$$= \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 + \cancel{2x} + 2h - \cancel{x^3} - \cancel{2x}}{h}$$

$$= \frac{3x^2h + 3xh^2 + h^3 + 2h}{h}$$

$$= \frac{\cancel{h}(3x^2 + 3xh + h^2 + 2)}{\cancel{h}}$$

$$= 3x^2 + 3xh + h^2 + 2$$

Now we'll take the limit as $h \to 0$.

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} (3x^2+3xh+h^2+2)$$

$$= 3x^2+3x\cdot 0+0^2+2$$

$$= 3x^2+2$$

* We took $h \to 0$ and treated x like a constant because x does not depend on h.

Section 1.3: Continuity

We have seen that their may or may not be a relationship between the quantities

$$\lim_{x\to c} f(x) \quad \text{and} \quad f(c).$$

One or the other (or both) may fail to exist. And even if both exist, they need not be equivalent.

We've also seen that for polynomials at least, that the limit at a point is the same as the function value at that point. Here, we explore this property that polynomials (and lots of other functions, but not all) share.

Definition: Continuity at a Point

Definition: A function f is continuous at a number c if

$$\lim_{x \to c} f(x) = f(c).$$

Note that three properties are contained in this statement:
(1) $f(c)$ is defined (i.e. c is in the domain of f),

(2) $\lim_{x \to c} f(x)$ exists, and

(3) the limit actually equals the function value.

If a function f is not continuous at c, we may say that f is **discontinuous** at c

Polynomials and Rational Functions

In the previous section, we saw that:

If P is any polynomial and c is any real number, then $\lim_{x \to c} P(x) = P(c)$, and

If R is any rational function and c is any number in the domain of R, then $\lim_{x \to c} R(x) = R(c)$.

Conclusion Theorem: Every rational function[1] is continuous at each number in its domain.

[1] Note that polynomials can be lumped in to the set of all rational functions.

Examples: Determine where each function is discontinuous.

(a) $g(t) = \dfrac{t^2 - 9}{t + 3}$

g is rational, so it's continuous at each number in its domain.

The domain will be all reals such that the denominator is not zero.

$t + 3 = 0 \implies t = -3$

g is discontinuous at $t = -3$ because $g(-3)$ is not defined.

As an aside note

$$\lim_{t\to -3} g(t) = \lim_{t\to -3} \frac{t^2-9}{t+3}$$

$$= \lim_{t\to -3} \frac{(t-3)\cancel{(t+3)}}{\cancel{t+3}}$$

$$= \lim_{t\to -3} (t-3) = -3-3 = -6$$

Despite the limit existing, g is discontinuous

@ $t=-3$.

(b) $f(x) = \begin{cases} 2x, & x < 1 \\ x^2 + 1, & 1 \le x < 2 \\ 3, & x \ge 2 \end{cases}$

The pieces are polynomials that are continuous independently.

On $(-\infty, 1)$, $f(x) = 2x$, so it's continuous on $(-\infty, 1)$

On $(1, 2)$, $f(x) = x^2 + 1$, so it's continuous on $(1, 2)$

On $(2, \infty)$, $f(x) = 3$, so it's continuous on $(2, \infty)$

We'll check for continuity @ $x = 1$.

① Does $f(1)$ exist? yes $f(1) = 1^2 + 1 = 2$

② Does $\lim_{x\to 1} f(x)$ exist?

$$\left.\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} 2x = 2\cdot 1 = 2 \\ \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} (x^2+1) = 1^2+1 = 2 \end{aligned}\right\} \text{yes } \lim_{x\to 1} f(x) = 2$$

③ Is $\lim_{x\to 1} f(x)$ the same as $f(1)$? Yes, f is continuous @ $x=1$

We'll check $x=2$:

① Is $f(2)$ defined? Yes $f(2)=3$

② Does $\lim_{x\to 2} f(x)$ exist?

$$\left.\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} (x^2+1) = 2^2+1 = 5 \\ \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} 3 = 3 \end{aligned}\right\}$$

No.

$\lim_{x\to 2} f(x)$ DNE

f is discontinuous @ $x = 2$.

Question

Determine whether f is continuous at 1 where $f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$

$f(1) = 2$

(a) No because $f(1)$ is not defined.

and

(b) Yes because all three conditions hold.

$\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$

(c) No because $\lim_{x \to 1} f(x)$ doesn't exist.

(d) No because f is piecewise defined.

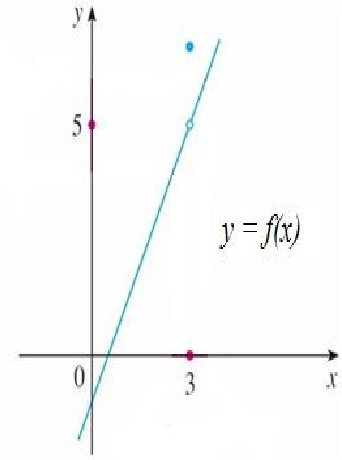
Removable and Jump Discontinuities

Definition: Let f be defined on an open interval containing c except possibly at c. If $\lim_{x\to c} f(x)$ exists, but f is discontinuous at c, then f has a **removable discontinuity** at c.

redefining the function @ c can make it continuous

Definition: If $\lim_{x\to c^-} f(x) = L_1$ and $\lim_{x\to c^+} f(x) = L_2$ where $L_1 \neq L_2$ (i.e. both one sided limits exist but are different), then f has a **jump discontinuity** at c.

Removable and Jump Discontinuities



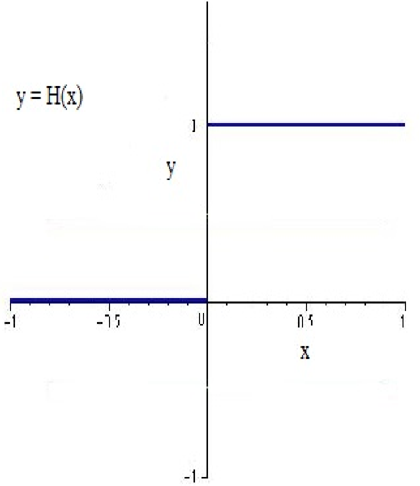


Figure: Example of a removable (left) discontinuity and a jump (right) discontinuity.

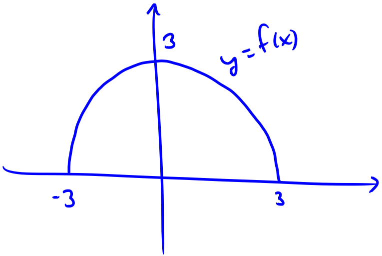
One Sided Continuity Example:

Consider the function $f(x) = \sqrt{9 - x^2}$. Plot a rough sketch of the graph of f, and determine its domain.

If $y = \sqrt{9-x^2} \Rightarrow y^2 = 9 - x^2 \Rightarrow x^2 + y^2 = 9$

top half of a circle

From the graph, the domain is $[-3, 3]$.



Alternatively, we require

$$9 - x^2 \geq 0$$

$$9 \geq x^2$$

$$\Rightarrow \quad \sqrt{9} \geq \sqrt{x^2} \quad \Longrightarrow \quad 3 \geq |x|$$

i.e. $\quad -3 \leq x \leq 3$.

Same conclusion, the domain is $[-3,3]$.

$f(x) = \sqrt{9 - x^2}$

Note that f is continuous on $-3 < x < 3$. What can be said about

$$\lim_{x \to -3} f(x) \qquad \text{or} \qquad \lim_{x \to 3} f(x)?$$

Not much. f is not defined on an open interval containing -3 or containing 3.

Continuity From the Left & Right

Definition: Let a function f be defined on an interval $[c,b)$. Then f is continuous from the right at c if

$$\lim_{x\to c^+} f(x) = f(c).$$

Let f be defined on an interval $(a,c]$. Then f is continuous from the left at c if

$$\lim_{x\to c^-} f(x) = f(c).$$

Example: $f(x) = \sqrt{9 - x^2}$

Compare $f(-3)$ and $\lim_{x \to -3^+} f(x)$. Is f continuous from the right at -3?

$$f(-3) = \sqrt{9 - (-3)^2} = \sqrt{9-9} = \sqrt{0} = 0.$$

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \sqrt{9 - x^2} = \sqrt{9 - (-3)^2} = 0$$

Since these both exist and are the same,

f is continuous from the right @ $x = -3$.

A Theorem on Continuous Functions

Theorem If f and g are continuous at c and for any constant k, the following are also continuous at c:

$$(i)\, f+g, \quad (ii)\, f-g, \quad (iii)\, kf, \quad (iv)\, fg, \quad \text{and} \quad (v)\, \frac{f}{g}, \text{ if } g(c) \neq 0.$$

In other words, if we combine continuous functions using addition, subtraction, multiplication, division, and using constant factors, the result is also continuous—provided of course that we don't introduce division by zero.

Questions

(1) **True or False** If f is continuous at 3 and g is continuous at 3, then it must be that

$$\lim_{x\to 3} f(x)g(x) = f(3)g(3).$$

yes since $f(x)\,g(x)$ is also cont. @ 3.

(2) **True or False** If $f(2) = 1$ and $g(2) = 7$, then it must be that

$$\lim_{x\to 2} \frac{f(x)}{g(x)} = \frac{1}{7}.$$

Consider the example

$g(x) = 7$ and $f(x) = \begin{cases} 1, & x \geq 2 \\ 0, & x < 2 \end{cases}$

Continuity on an Interval

Definition A function is continuous on an interval (a, b) if it is continuous at each point in (a, b). A function is continuous on an interval such as $(a, b]$ or $[a, b)$ or $[a, b]$ provided it is continuous on (a, b) and has one sided continuity at each included end point.

Graphically speaking, if $f(x)$ is continuous on an interval (a, b), then the curve $y = f(x)$ will have no holes or gaps.