Jan. 26 Math 2254H sec 015H Spring 2015

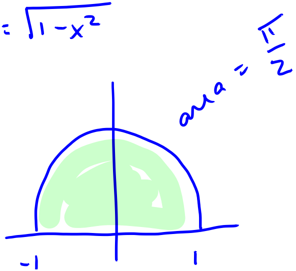
Section 7.3 Trigonometric Substitution

Compare the two integrals:

$$\int x\sqrt{1-x^2}\,dx, \quad \text{and} \quad \int \sqrt{1-x^2}\,dx.$$

Using some simple geometry

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{\pi}{2}.$$



It's clear from geometry, but **not directly** from any algebra where a factor of π comes from!

A different sort of substitution

$$\int \sqrt{1-x^2}\, dx$$

Consider the new variable θ defined by $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and

$$x = \sin\theta, \quad \text{so that} \quad dx = \cos\theta\, d\theta$$

$$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta|$$

$$= \cos\theta \quad \text{for} \quad -\pi/2 < \theta < \pi/2$$

$$\int \sqrt{1-x^2}\,dx = \int \cos\theta \cos\theta \, d\theta$$

$$= \int \cos^2\theta \, d\theta$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$= \int \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right) d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C$$

$$= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$$

$$= \frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{1-x^2} + C$$