January 26 Math 2306 sec 58 Spring 2016

Section 4: First Order Equations: Linear

A first order linear equation has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$ the equation is called **homogeneous.** Otherwise it is called **nonhomogeneous.**

Provided $a_1(x) \neq 0$ on the interval I of definition of a solution, we can write the **standard form** of the equation

$$\frac{dy}{dx} + P(x)y = f(x).$$

$P(x) = a_0/a_1$
$f(x) = g/a_1$

We'll be interested in equations (and intervals I) for which P and f are continuous on I.

Solutions (the General Solution)

$$\frac{dy}{dx} + P(x)y = f(x).$$

It turns out the solution will always have a basic form of $y = y_c + y_p$ where

- y_c is called the **complementary** solution and would solve the problem
$$\frac{dy}{dx} + P(x)y = 0$$
(called the associated homogeneous equation), and
- y_p is called the **particular** solution, and is heavily influenced by the function $f(x)$.

The cool thing is that our solution method will get both parts in one process—we won't get this benefit with higher order equations!

Motivating Example

The eqn. isn't in standard form. But we'll leave it as is.

$$x^2\frac{dy}{dx}+2xy=e^x$$

Note: the left is a product rule.

$$\frac{d}{dx}\left(x^2y\right) = x^2\frac{dy}{dx}+2xy$$

Hence our equation is actually

$$\frac{d}{dx}\left(x^2y\right) = e^x$$

We will undo the derivative on the left.

$$\int \frac{d}{dx}\left(x^2 y\right) dx = \int e^x \, dx \quad \leftarrow \text{integrate}$$

$$x^2 y = e^x + C$$

Divide by x^2 to get our solution y.

$$\boxed{y = \frac{e^x + C}{x}}$$

Derivation of Solution via Integrating Factor

Solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

We'd like to force the left side to be a product rule. We'll multiply by some function $\mu(x)$ to achieve this.

Multiply by $\mu(x)$

$$\mu(x)\frac{dy}{dx} + P(x)\mu(x)y = \mu(x)f(x)$$

Note: $\frac{d}{dx}\left[\mu(x)y\right] = \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y$

Compare the left side of the eqn. with the product rule

$$\mu\frac{dy}{dx} + P(x)\mu y = \mu\frac{dy}{dx} + \frac{d\mu}{dx}y$$

This requires $P(x)\mu = \frac{d\mu}{dx}$

i.e. $$\frac{d\mu}{dx} = P(x)\mu$$

a separable equation!

Solve
$$\frac{1}{\mu}\frac{d\mu}{dx} = P(x) \Rightarrow \frac{1}{\mu}\,d\mu = P(x)\,dx$$

$$\int \frac{1}{\mu}\,d\mu = \int P(x)\,dx \Rightarrow \ln\mu = \int P(x)\,dx$$

Exponentiate to get
$$\boxed{\mu = e^{\int P(x)\,dx}}$$
called an integrating factor.

Now, return to the ODE

$$\mu\frac{dy}{dx} + \mu P y = \mu(x) f(x)$$
$$\frac{d}{dx}\left[\mu y\right] = \mu(x) f(x)$$

$$\int \frac{d}{dx}[\mu y]\,dx = \int \mu(x) f(x)\,dx$$

$$\mu y = \int \mu(x) f(x)\,dx + C$$

Solve for y

$$y = \underbrace{\mu^{-1}\int \mu(x) f(x)\,dx}_{y_p} + \underbrace{C\mu^{-1}}_{y_c}$$

General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx} f(x)\,dx + C\right)$$

Solve the ODE

$$x^2\frac{dy}{dx}+2xy=e^x$$

Standard form

$$\Longrightarrow \quad \frac{dy}{dx}+\frac{2}{x}y=\frac{e^x}{x^2}$$

$P(x)=\frac{2}{x}$ $\quad \mu = e^{\int P(x)\,dx}$

$$\int P(x)\,dx = \int \frac{2}{x}\,dx = 2\ln|x| = \ln x^2$$

$$\mu = e^{\ln x^2} = x^2$$

$*\quad r\ln x = \ln x^r$

Mult. eqn by μ

$$x^2\frac{dy}{dx}+2xy=\frac{e^x}{x^2}\cdot x^2$$

$$\frac{d}{dx}\left[x^2 y\right] = e^x$$

$$\int \frac{d}{dx}\left[x^2 y\right] dx = \int e^x \, dx$$

$$x^2 y = e^x + C$$

The solution to the ODE is then

$$y = \frac{e^x + C}{x^2}.$$

Solve the ODE

$$\frac{dy}{dx} + y = 3xe^{-x}$$

The ODE is in Standard form:

$P(x) = 1$ $\int P(x)\,dx = \int dx = x$

$\mu = e^{\int P(x)dx} = e^{x}$

Mult. by μ $e^{x}\frac{dy}{dx} + e^{x}y = 3xe^{-x} \cdot e^{x}$

$$\frac{d}{dx}\left[e^{x}y\right] = 3x$$

$$\int \frac{d}{dx}\left[e^{x} y\right] dx = \int 3x \; dx$$

$$e^{x} y = 3\frac{x^2}{2} + C$$

$$y = \frac{\frac{3}{2}x^2 + C}{e^{x}} = \frac{3}{2}x^2 e^{-x} + C e^{-x}$$

The solution is $\boxed{y = \frac{3}{2}x^2 e^{-x} + C e^{-x}}$.

Solve the IVP

$$x\frac{dy}{dx} - y = 2x^2, \quad y(1) = 5$$

Since $x_0 = 1 > 0$, we can assume that $x > 0$.

Put ODE in standard form

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{2x^2}{x} = 2x$$

$$P(x) = \frac{-1}{x} \qquad \int P(x)\,dx = \int \frac{-1}{x}\,dx = -\ln|x| = \ln x^{-1}$$

$$\mu = e^{\int P(x)\,dx} = e^{\ln x^{-1}} = x^{-1}$$

Mult by μ $\quad x^{-1}\frac{dy}{dx} - x^{-1}\cdot\frac{1}{x}y = 2x\cdot x^{-1}$

$$\frac{d}{dx}\left[x^{-1}y\right] = 2$$

$$\int \frac{d}{dx}\left[x^{-1}y\right]dx = \int 2\,dx$$

$$x^{-1}y = 2x + C$$

$$y = \frac{2x+C}{x^{-1}} = (2x+C)x = 2x^2 + Cx$$

$$y = 2x^2 + Cx$$

Apply $y(1) = 5$ $\qquad y(1) = 2\cdot 1^2 + C\cdot 1 = 5 \Rightarrow C = 3$

$$\boxed{\begin{array}{l}\text{The solution to the IVP is} \\[2em] \qquad\qquad y = 2x^2 + 3x.\end{array}}$$

Solve the IVP

We can assume that $t < 0$.

$$\frac{dy}{dt} + \frac{4}{t}y = \frac{e^t}{t^3}, \quad y(-1) = 0$$

The ODE is in standard form, $P(t) = \frac{4}{t}$

$$\int P(t)\,dt = \int \frac{4}{t}\,dt = 4\ln|t| = \ln t^4$$

$*$ 4 is even

$|t| = -t$ for $t<0$

$$\mu = e^{\int P(t)\,dt} = e^{\ln t^4} = t^4$$

e.g. $5\ln|t| = (-t)^5$

$= -t^5$

$$t^4 \frac{dy}{dt} + \frac{4}{t} t^4 y = \frac{e^t}{t^3} t^4$$

$$\frac{d}{dt}\left[t^4 y\right] = te^t$$

Use parts

$u = t \quad du = dt$

$v = e^t \quad dv = e^t\,dt$

$$\int \frac{d}{dt}\left[t^4 y\right] dt = \int te^t\,dt$$

$$t^4 y = te^t - \int e^t\,dt$$

$$= te^t - e^t + C$$

$$\Rightarrow \quad y = \frac{te^t - e^t + C}{t^4}$$

Apply $y(-1)=0$

$$y(-1) = \frac{-1e^{-1} - e^{-1} + C}{(-1)^4} = -2e^{-1} + C = 0$$

$$\Rightarrow \quad C = 2e^{-1}$$

The solution to the IVP is

$$y = \frac{te^{t} - e^{t} + 2e^{-1}}{t^4}.$$

Steady and Transient States

For some linear equations, the term y_c decays as x (or t) grows. For example

$$\frac{dy}{dx} + y = 3xe^{-x} \quad \text{has solution} \quad y = \frac{3}{2}x^2 + Ce^{-x}.$$

$$\text{Here,} \quad y_p = \frac{3}{2}x^2 \quad \text{and} \quad y_c = Ce^{-x}.$$

Note $\lim_{x\to\infty} Ce^{-x} = 0$

Such a decaying complementary solution is called a **transient state.**

The corresponding particular solution is called a **steady state.**

Section 5: First Order Equations Models and Applications

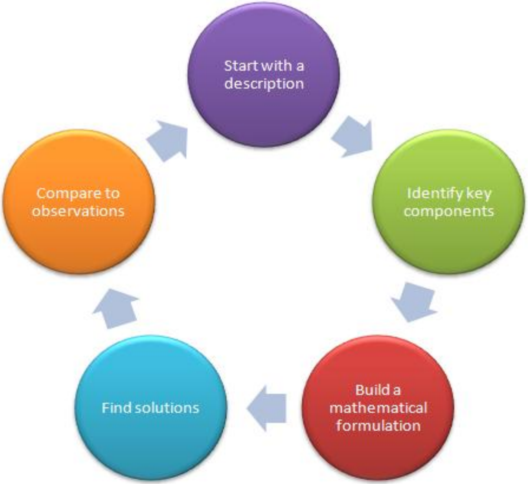

Figure: Mathematical Models give Rise to Differential Equations

Population Dynamics

A population of dwarf rabbits grows at a rate proportional to the current population. In 2011, there were 58 rabbits. In 2012, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2021.

Let $P(t)$ be the population (or population density) of rabbits at time t.

The rate of change of P is $\frac{dP}{dt}$

The first sentence says $\frac{dP}{dt} \propto P$

i.e. $\frac{dP}{dt} = kP$ for some constant k.

This is an ODE. If we take t in years and let $t=0$ in 2011. Our statement gives

$$P(0) = 58 \quad \text{and} \quad P(1) = 89$$

We'll finish this on Feb. 2.