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Section 4: First Order Equations: Linear

A first order linear equation has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$ the equation is called **homogeneous.** Otherwise it is called **nonhomogeneous.**

Provided $a_1(x) \neq 0$ on the interval I of definition of a solution, we can write the **standard form** of the equation

$P(x) = \frac{a_0(x)}{a_1(x)}$

$$\frac{dy}{dx} + P(x)y = f(x).$$

$f(x) = \frac{g(x)}{a_1(x)}$

We'll be interested in equations (and intervals I) for which P and f are continuous on I.

Solutions (the General Solution)

$$\frac{dy}{dx} + P(x)y = f(x).$$

It turns out the solution will always have a basic form of $y = y_c + y_p$ where

- y_c is called the **complementary** solution and would solve the problem
$$\frac{dy}{dx} + P(x)y = 0$$
(called the associated homogeneous equation), and
- y_p is called the **particular** solution, and is heavily influenced by the function $f(x)$.

The cool thing is that our solution method will get both parts in one process—we won't get this benefit with higher order equations!

Motivating Example

$$x^2\frac{dy}{dx}+2xy=e^x \quad , \quad x>0$$

This eqn. is not in standard form, but we'll leave it as is.

Note $\frac{d}{dx}\left[x^2y\right] = x^2\frac{dy}{dx} + 2xy$

Hence our equation is $\frac{d}{dx}\left[x^2y\right] = e^x$

We wish to find y:

$$\int \frac{d}{dx}\left[x^2y\right]dx = \int e^x\,dx$$

$$x^2 y = e^x + C$$

Divide out x^2

$$y = \frac{e^x + C}{x^2}$$

$$y = \underbrace{\frac{C}{x^2}}_{y_c} + \underbrace{\frac{e^x}{x^2}}_{y_p}$$

Derivation of Solution via Integrating Factor

Solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

We wish to make the left side collapse as a product rule. We'll multiply the equation by some function $\mu(x)$ to accomplish this.

Mult. by μ: $\quad \mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)f(x)$

Note: $\frac{d}{dx}\left[\mu(x)y\right] = \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y$

Comparing this to the left side of the ODE, if

$$\mu P y = \frac{d\mu}{dx}y$$

Then the left side will be a product rule.

This requires $\frac{d\mu}{dx} = P(x)\mu$ a separable equation.

Solve this for μ.

$$\frac{1}{\mu}\frac{d\mu}{dx} = P(x) \Rightarrow \frac{1}{\mu}\, d\mu = P(x)\, dx$$

$$\int \frac{1}{\mu}\, d\mu = \int P(x)\, dx \Rightarrow \ln \mu = \int P(x)\, dx$$

Exponentiate

$$\boxed{\mu = e^{\int P(x)\, dx}}$$

This is called an integrating factor.

Back to the equation

$$\mu \frac{dy}{dx} + \mu P y = \mu(x) f(x)$$

$$\frac{d}{dx}\left[\mu y\right] = \mu(x) f(x)$$

$$\int \frac{d}{dx}\left[\mu y\right] dx = \int \mu(x) f(x)\, dx$$

$$\mu y = \int \mu(x) f(x)\, dx + C$$

$$y = \underbrace{\mu^{-1} \int \mu(x) f(x)\, dx}_{y_p} + \underbrace{C \mu^{-1}}_{y_c}$$

General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx}f(x)\,dx + C\right)$$

Solve the ODE

$$x^2\frac{dy}{dx}+2xy=e^x \quad , x>0$$

Put the ODE in standard form.

$$\frac{dy}{dx}+\frac{2}{x}y=\frac{e^x}{x^2}$$

$$P(x)=\frac{2}{x} \qquad \int P(x)\,dx=\int\frac{2}{x}\,dx=2\ln x=\ln x^2$$

$$* \quad r\ln x=\ln x^r$$

$$\mu=e^{\int P(x)\,dx}=e^{\ln x^2}=x^2$$

Mult. by μ $$x^2\frac{dy}{dx} + x^2\cdot\frac{2}{x}y = x^2\cdot\frac{e^x}{x^2}$$

$$\frac{d}{dx}\left[x^2y\right] = e^x$$

$$\int \frac{d}{dx}\left[x^2y\right]dx = \int e^x\,dx$$

$$x^2y = e^x + C$$

$$\boxed{y = \frac{e^x+C}{x^2}}$$

a one parameter family of solutions

Solve the ODE

$$\frac{dy}{dx}+y=3xe^{-x}$$

This is in standard form.

$P(x) = 1$

$\int P(x)\,dx = \int dx = x$

$\mu = e^{\int P(x)\,dx} = e^{x}$

Mult. by μ: $e^{x}\frac{dy}{dx} + e^{x}y = 3xe^{-x}\cdot e^{x}$

$\frac{d}{dx}\left[e^{x}y\right] = 3x$

$$\int \frac{d}{dx}\left[e^x y\right] dx = \int 3x \, dx$$

$$e^x y = \frac{3x^2}{2} + C$$

$$y = \frac{\frac{3x^2}{2} + C}{e^x} = \frac{3}{2}x^2 e^{-x} + Ce^{-x}$$

$$\boxed{y = \frac{3}{2}x^2 e^{-x} + Ce^{-x}}$$

Solve the IVP

$$x\frac{dy}{dx} - y = 2x^2, \quad y(1) = 5$$

Our initial condition is @ $x_0 = 1$. We may assume that $x > 0$.

Standard form:

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{2x^2}{x} = 2x$$

$$P(x) = \frac{-1}{x} \qquad \int P(x)\,dx = \int \frac{-1}{x}\,dx = -\ln x = \ln x^{-1}$$

$$\mu = e^{\int P(x)dx} = e^{\ln x^{-1}} = x^{-1}$$

Mult. by μ: $$x^{-1}\frac{dy}{dx} - x^{-1}\cdot\frac{1}{x}\,y = 2x\cdot x^{-1}$$

$$\frac{d}{dx}\left[x^{-1}y\right] = 2$$

$$\int \frac{d}{dx}\left[x^{-1}y\right]dx = \int 2\,dx$$

$$x^{-1}y = 2x + C$$

$$y = \frac{2x+C}{x^{-1}} = (2x+C)\,x = 2x^2 + Cx$$

$$y = 2x^2 + Cx$$

This is the general solution to the ODE.

Impose the condition $y(1) = 5$

$$y(1) = 2 \cdot 1^2 + C \cdot 1 = 5 \implies C = 5 - 2 = 3.$$

$$\boxed{\begin{array}{l}\text{The solution to the IVP is} \\ \\ \qquad y = 2x^2 + 3x.\end{array}}$$

Solve the IVP

$$\frac{dy}{dt}+\frac{4}{t}y=\frac{e^t}{t^3}, \quad y(-1)=0$$

The initial condition is given at $t_0=-1$.

We may assume that $t<0$.

$P(t)=\frac{4}{t}$ $\quad \int P(t)\,dt=\int \frac{4}{t}\,dt = 4\ln|t| = \ln t^4$

$$\mu = e^{\int P(t)\,dt} = e^{\ln t^4} = t^4$$

$$t^4\frac{dy}{dt}+t^4\cdot\frac{4}{t}\,y = t^4\cdot\frac{e^t}{t^3}$$

$$\frac{d}{dt}\left[t^4 y\right] = te^t$$

$$\int \frac{d}{dt}\left[t^4 y\right] dt = \int te^t \, dt$$

Use parts

$u = t \quad du = dt$

$v = e^t \quad dv = e^t dt$

$$t^4 y = te^t - \int e^t \, dt$$

$$= te^t - e^t + C$$

$$\Rightarrow \quad y = \frac{te^t - e^t + C}{t^4}$$

Apply $y(-1)=0$

$$y(-1) = \frac{-1e^{-1} - e^{-1} + C}{(-1)^4} = -2e^{-1} + C = 0$$

$$C = 2e^{-1}$$

The solution of the IVP is

$$\boxed{y = \frac{te^t - e^t + 2e^{-1}}{t^4}.}$$

Steady and Transient States

For some linear equations, the term y_c decays as x (or t) grows. For example

$$\frac{dy}{dx} + y = 3xe^{-x} \quad \text{has solution} \quad y = \frac{3}{2}x^2 + Ce^{-x}.$$

$$\text{Here,} \quad y_p = \frac{3}{2}x^2 \quad \text{and} \quad y_c = Ce^{-x}.$$

Note $\lim_{x\to\infty} Ce^{-x} = 0$

Such a decaying complementary solution is called a **transient state.**

The corresponding particular solution is called a **steady state.**