January 26 Math 2335 sec 51 Spring 2016

Section 2.2: Errors (definitions, sources, and examples)

Definition: The **error** in a computed quantity is

$$\text{Error} \quad = \quad \text{true value} \quad - \quad \text{approximated value.}$$

The **relative error** in a computed quantity is

$$\text{Relative Error} \quad = \quad \frac{\text{true value} \quad - \quad \text{approximated value}}{\text{true value}}.$$

Notation

Suppose we wish to compute a quantity x. We will use the following notation: let

$$
\begin{aligned}
x_T \quad &- \quad \text{denote the true value} \\
x_A \quad &- \quad \text{denote the approximated value}
\end{aligned}
$$

the errors are denoted

$$
\begin{aligned}
\mathrm{Err}(x_A) &= x_T - x_A \\
\mathrm{Rel}(x_A) &= \frac{x_T - x_A}{x_T} = \frac{\mathrm{Err}(x_A)}{x_T}
\end{aligned}
$$

Example

Suppose we take the *true* value of π to be $x_T = \pi = 3.14159265$. Determine the error and the relative error when

$$x_A = \frac{22}{7}$$

is used to approximate π.

$$\mathrm{Err}\left(\frac{22}{7}\right) = 3.14159265 - \frac{22}{7} \doteq -0.00126449$$

$$\mathrm{Rel}\left(\frac{22}{7}\right) = \frac{3.14159265 - \frac{22}{7}}{3.14159265} \doteq -0.00040250$$

Example

Suppose we take the *true* value of ln(3) to be $x_T = \ln(3) = 1.0986123$. Determine the error and the relative error when

$$x_A = \frac{78}{71}$$

is used to approximate ln(3).

$$\mathrm{Err}\left(\frac{78}{71}\right) = 1.0986123 - \frac{78}{71} \doteq 0.00002075$$

$$\mathrm{Rel}\left(\frac{78}{71}\right) = \frac{\mathrm{Err}\left(\frac{78}{71}\right)}{1.0986123} \doteq 0.00001889$$

Example

We used $p_1(x)$ for $f(x) = \sqrt{x}$ centered at $a = 4$ to approximate $\sqrt{4.1}$. Identify x_T, x_A, and find the error and relative error.

$x_T = \sqrt{4.1}$ (From Jan. 19) $x_A = p_1(4.1) = 2.025$

$$\text{Err}(2.025) = \sqrt{4.1} - 2.025 \doteq -0.00015430$$

$$\text{Rel}(2.025) = \frac{\text{Err}(2.025)}{\sqrt{4.1}} \doteq -0.00007620$$

Why consider relative error...?

Relative error has more *intrinsic* meaning. It is also a unit-less measure. Consider the following scenarios:

Example 1: A colony of bacteria is known to reproduce with a doubling time of 5 hours. The doubling time recorded in the laboratory is 298 minutes (4 hrs 58 min). The error and relative error are

$$\mathrm{Err}(298) = 2min, \quad \text{and} \quad \mathrm{Rel}(298) = 0.0066\bar{6}.$$

Example 2: It takes 6 min. 40 sec. for half a sample of N_2O_5 to decompose (into $NO_2 + O_2$). The time recorded in the lab is 280 sec. The error and relative error are

$$\mathrm{Err}(280) = 2min, \quad \text{and} \quad \mathrm{Rel}(280) = 0.3.$$

Significant Digits

An idea related to relative error is significant digits. An informal definition:

Informal Definition: For an approximation x_A, the number of its leading digits (count from left to right starting with the first nonzero digit) that are correct relative to the corresponding digits of the true value x_T is called the *number of significant digits* in x_A.

Examples: (a) $x_A = \frac{22}{7} = 3.142857$ has three significant digits as an approximation to π

(b) $x_A = 34.1456$ has two significant digits as an approximation to $x_T = 34.2355$

(c) $x_A = 2$ has one significant digit as an approximation to $x_T = e$.

Sources of Error

- Modeling errors (simplifications made)
- Measurement errors (finite precision instruments and human error)
- Machine errors (rounding/truncating/change of base)
- Mathematical approximation (e.g. using p_n to approximate f)

Loss of Significance

Loss of significant digits can occur in machine computations when number that are very close are **subtracted**. Consider the following example:

We wish to compute $\sqrt{101} - \sqrt{100}$ on a six digit calculator.

$$\sqrt{100} = 10.0000 \quad \text{and} \quad \sqrt{101} = 10.0499$$

to six digits of accuracy. The difference calculated is

$$x_A = \sqrt{101} - \sqrt{100} \doteq 0.0499000$$

The true value to six digits is $x_T = 0.0498756$. So x_A only has two significant digits!

Strategies to Avoid Loss of Significance

A general rule of thumb is to avoid subtraction!

How this can be done depends on the problem. Let's consider several examples.

Example: Using Algebra

Let $f(x) = \sqrt{x+1} - \sqrt{x}$. For x large, find an alternative expression for $f(x)$ that avoids subtraction.

$\sqrt{x+1} + \sqrt{x}$ is the conjugate expression

$$f(x) = \left(\sqrt{x+1} - \sqrt{x}\right) \cdot \left(\frac{\sqrt{x+1} + \sqrt{x}}{\sqrt{x+1} + \sqrt{x}}\right)$$

$$= \frac{x+1 - x}{\sqrt{x+1} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x+1} + \sqrt{x}}$$

$$f(x) = \frac{1}{\sqrt{x+1} + \sqrt{x}}$$

This expression requires no subtraction.

Example: Using Identities

Let

$$g(x) = \frac{1 - \cos x}{x^2}.$$

Several values of g for x close to zero were computed on a 10 digit calculator as shown in the table.

x	computed	true value
0.1	0.4995834700	0.4995834722
0.01	0.4999960000	0.4999958333
0.001	0.5000000000	0.4999999583
0.0001	0.5000000000	0.4999999996
0.00001	0.0	0.5000000000

Example Continued...

Use the identity $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$ to express $g(x)$ in a way that avoids subtraction.

$$g(x) = \frac{1-\cos x}{x^2}$$

From $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$

$$2\sin^2\theta = 1 - \cos 2\theta$$

Take $2\theta = x \Rightarrow \theta = \frac{x}{2}$

so $1 - \cos x = 2\sin^2\frac{x}{2}$

$$g(x) = \frac{2\sin^2\left(\frac{x}{2}\right)}{x^2}$$

Here's an equivalent express that avoids

subtraction.

A second approach

Again consider $g(x) = \frac{1-\cos x}{x^2}$. Use the Taylor polynomial with remainder

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + R_4(x), \quad \text{where} \quad R_4(x) = \frac{-x^6}{6!}\cos(c_x)$$

to find a Taylor polynomial approximation to g, and bound the error for $|x| \le 0.1$.

$$g(x) = \frac{1-\cos x}{x^2} = \frac{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}\cos(c_x)\right)}{x^2}$$

$$= \frac{\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}\cos(c_x)}{x^2}$$

$$= \frac{1}{2}-\frac{x^2}{24}+\frac{x^4}{720}\cos(c_x)$$

For some c_x between zero and x

Our remainder for g is $R_2(x) = \frac{x^4}{720}\cos(c_x)$

For $|x| \leq 0.1$, $|x|^4 \leq (0.1)^4 = 10^{-4}$

For $|c_x| \leq 0.1$ $|\cos(c_x)| \leq 1$

$$|R_2(x)| = \left|\frac{x^4}{720}\cos(c_x)\right| \leq \frac{10^{-4}}{720}\cdot 1$$

$$\doteq 0.00000014$$