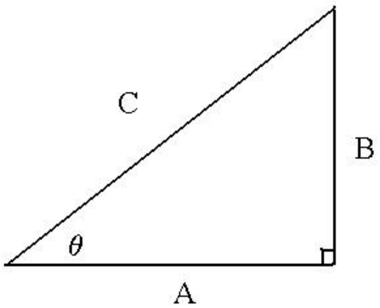
Jan. 27 Math 2254H sec 015H Spring 2015

Section 7.3 Trigonometric Substitution

May be useful when we see terms in an integral that look like (a is a constant)

$$\sqrt{a^2 - x^2}, \quad \sqrt{x^2 - a^2}, \quad \text{or} \quad \sqrt{a^2 + x^2}.$$



$$C = \sqrt{A^2 + B^2}$$

$$A = \sqrt{C^2 - B^2}$$

$$B = \sqrt{C^2 - A^2}$$

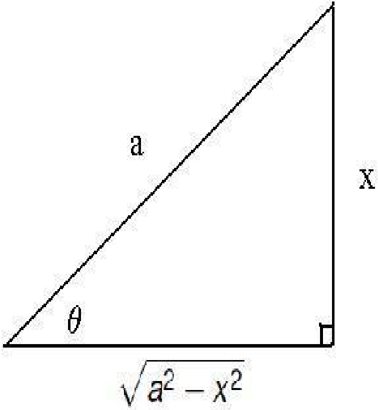
Figure: Trig Substitution Motivated by the Pythagorean Theorem

Substitution for the form $\sqrt{a^2 - x^2}$

$$x = a\sin\theta$$

$$dx = a\cos\theta\, d\theta$$

$$\sqrt{a^2 - x^2} = a\cos\theta$$

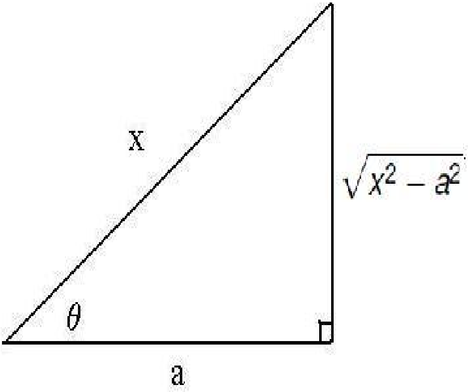


Substitution for the form $\sqrt{x^2 - a^2}$

$$x = a \sec\theta$$

$$dx = a \sec\theta \tan\theta \, d\theta$$

$$\sqrt{x^2 - a^2} = a \tan\theta$$

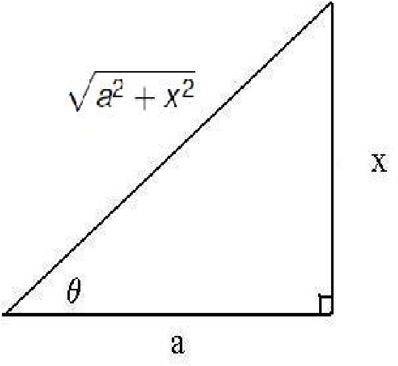


Substitution for the form $\sqrt{a^2 + x^2}$

$$x = a\tan\theta$$

$$dx = a\sec^2\theta\, d\theta$$

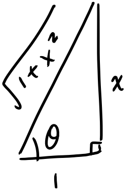
$$\sqrt{x^2 + a^2} = a\sec\theta$$



We'll assume that θ is in an appropriate interval–e.g. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for the substitution $x = a\tan\theta$

Evaluate The Integral

(a) $\displaystyle\int \frac{dx}{1+x^2} = \int \frac{dx}{\left(\sqrt{1+x^2}\right)^2}$



$\tan\theta = \dfrac{x}{1}$

$x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

$\sqrt{1+x^2} = \sec\theta$

$\displaystyle = \int \frac{\sec^2\theta \, d\theta}{(\sec\theta)^2}$

$\displaystyle = \int d\theta$

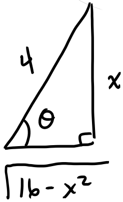
Note $\sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta}$

$$= \theta + C$$

$$= \tan^{-1} x + C$$

Evaluate The Integral

(b) $$\int \frac{\sqrt{16-x^2}}{x^2}\,dx$$



$$= \int \frac{4\cos\theta \cdot 4\cos\theta\, d\theta}{(4\sin\theta)^2}$$

$$x = 4\sin\theta$$

$$dx = 4\cos\theta\, d\theta$$

$$\cos\theta = \frac{\sqrt{16-x^2}}{4}$$

$$4\cos\theta = \sqrt{16-x^2}$$

$$= \int \cot^2\theta\, d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= \int \frac{1-\sin^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int \left(\frac{1}{\sin^2\theta} - 1\right)d\theta$$

$$= \int (\csc^2\theta - 1)\,d\theta$$

$$= -\cot\theta - \theta + C$$

$$\cot\theta = \frac{\sqrt{16-x^2}}{x}$$

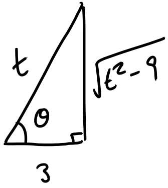
$$x = 4\sin\theta$$

$$\sin\theta = \frac{x}{4}$$

$$\theta = \sin^{-1}\left(\frac{x}{4}\right)$$

$$\int \frac{\sqrt{16-x^2}}{x^2}\,dx = -\frac{\sqrt{16-x^2}}{x} - \sin^{-1}\left(\frac{x}{4}\right) + C$$

Evaluate The Integral



(c) $\displaystyle\int \frac{dt}{t^2\sqrt{t^2-9}}$

$t = 3\sec\theta$

$dt = 3\sec\theta\tan\theta\, d\theta$

$\tan\theta = \dfrac{\sqrt{t^2-9}}{3}$

$\sqrt{t^2-9} = 3\tan\theta$

$$= \int \frac{3\sec\theta\tan\theta\, d\theta}{(3\sec\theta)^2\, 3\tan\theta}$$

$$= \frac{1}{9}\int \frac{d\theta}{\sec\theta}$$

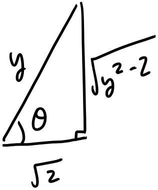
$$= \frac{1}{9}\int \cos\theta\, d\theta$$

$$= \frac{1}{9} \sin\theta + C$$

$$= \frac{1}{9} \frac{\sqrt{t^2-9}}{t} + C$$

Evaluate The Integral (other half integer roots)

(d) $$\int \frac{dy}{(y^2-2)^{3/2}} = \int \frac{dy}{\left(\sqrt{y^2-2}\right)^3}$$



$$= \int \frac{\sqrt{2}\tan\theta\sec\theta\,d\theta}{(\sqrt{2}\tan\theta)^3}$$

$$y = \sqrt{2}\sec\theta$$

$$= \frac{1}{2}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$dy = \sqrt{2}\sec\theta\tan\theta\,d\theta$$

$$\sqrt{y^2-2} = \sqrt{2}\tan\theta$$

$$= \frac{1}{2}\int \frac{\cos^2\theta}{\cos\theta\sin^2\theta}\,d\theta$$

$$= \frac{1}{2} \int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{2} \int \cot\theta \csc\theta \, d\theta$$

$$= -\frac{1}{2} \csc\theta + C$$

$$= -\frac{1}{2}\left(\frac{y}{\sqrt{y^2-2}}\right) + C = \frac{-y}{2\sqrt{y^2-2}} + C$$

Evaluate The Integral (completing the square)

(e) $\displaystyle\int_5^9 \frac{dt}{\sqrt{t^2-6t+13}}$

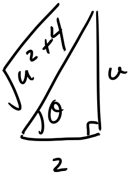
$$t^2-6t+13 = t^2-6t+9+4$$
$$= (t-3)^2+4$$

$$\int \frac{dt}{\sqrt{t^2-6t+13}}$$

$$= \int \frac{dt}{\sqrt{(t-3)^2+4}}$$

$u = t-3, \quad du = dt$

$$= \int \frac{du}{\sqrt{u^2+4}}$$



$u = 2\tan\theta$

$du = 2\sec^2\theta \, d\theta$

$\sqrt{u^2+4} = 2\sec\theta$

$$= \int \frac{2\sec^2\theta \, d\theta}{2\sec\theta} = \int \sec\theta \, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{u^2+4}}{2} + \frac{u}{2}\right| + C$$

$$= \ln\left|\frac{\sqrt{(t-3)^2+4}}{2} + \frac{t-3}{2}\right| + C = \ln\left|\sqrt{(t-3)^2+4} + t-3\right| + K$$

$$\int_5^9 \frac{dx}{\sqrt{t^2-6t+13}} = \ln\left|\sqrt{(t-3)^2+4} + t-3\right| \Big|_5^9 = \ln(\sqrt{40}+6) - \ln(\sqrt{8}+2)$$