January 27 Math 3260 sec. 55 Spring 2020

Section 1.4: The Matrix Equation $A\mathbf{x} = \mathbf{b}$.

For $m \times n$ matrix A and vector $\mathbf{x}$ in $\mathbb{R}^n$, we defined the product

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

We saw that the matrix equation $A\mathbf{x} = \mathbf{b}$ is equivalent to the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which is equivalent to the linear system whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}].$$

Corollary

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

In other words, the corresponding linear system is consistent if and only if $\mathbf{b}$ is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

Example

Characterize the set of all vectors $\mathbf{b} = (b_1, b_2, b_3)$ such that $A\mathbf{x} = \mathbf{b}$ has a solution where

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}.$$

We set up an augmented matrix and did row operations to get

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \text{ ref} \longrightarrow \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & -2b_1 + b_2 - 2b_3 \end{bmatrix}$$

Vector Parametric Form

The previous system was consistent if and only if **b** solves the linear system

$$-2b_1 + b_2 - 2b_3 = 0.$$

This has solution set in **parametric form**[1]

$$\begin{array}{rcl} b_1 & = & \frac{1}{2}b_2 - b_3 \\ b_2, b_3 & - & \text{free} \end{array}$$

This solution can be stated in **vector parametric form**

$$\mathbf{b} = b_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

[1] See the lecture slides from January 13

Theorem (first in a string of equivalency theorems)

Let A be an $m \times n$ matrix. Then the following are logically equivalent (i.e. they are either all true or are all false).

(a) For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(b) Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

(d) A has a pivot position in every row.

(Note that statement (d) is about the *coefficient* matrix A, not about an augmented matrix $[A \quad \mathbf{b}]$.)

Computing $A\mathbf{x}$

We can use a *row-vector* dot product rule. The i^{th} entry is $A\mathbf{x}$ is the sum of products of corresponding entries from row i of A with those of $\mathbf{x}$. For example

$$\begin{bmatrix} 1 & 0 & -3 \\ -2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \end{bmatrix}$$

2×3 $\quad 3\times 1$ $\quad \to 2\times 1$

1st entry: $1(2)+0(1)+(-3)(-1)$

2nd entry: $-2(2)+(-1)(1)+4(-1)$

$$\begin{bmatrix} 2 & 4 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

3×2 $\quad 2 \times 1$

$\downarrow$

3×1

1st: $2(-3) + 4(2)$

2nd: $-1(-3) + 1(2)$

3rd: $0(-3) + 3(2)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

3×3 $\quad 3 \times 1$

$\downarrow$

3×1

$1(x_1) + 0(x_2) + 0(x_3)$

$0(x_1) + 1(x_2) + 0(x_3)$

$0(x_1) + 0(x_2) + 1(x_3)$

Identity Matrix

We'll call an $n \times n$ matrix with 1's on the diagonal and 0's everywhere else—i.e. one that looks like

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

the $n \times n$ **identity** matrix and denote it by I_n. (We'll drop the subscript if it's obvious from the context.)

This matrix has the property that for each $\mathbf{x}$ in $\mathbb{R}^n$

$$I_n\mathbf{x} = \mathbf{x}.$$

Theorem: Properties of the Matrix Product

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is any scalar, then

(a) $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$, and

(b) $A(c\mathbf{u}) = cA\mathbf{u}$.

Section 1.5: Solution Sets of Linear Systems

Definition A linear system is said to be **homogeneous** if it can be written in the form

$$A\mathbf{x} = \mathbf{0}$$

for some $m \times n$ matrix A and where $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.

Theorem: A homogeneous system $A\mathbf{x} = \mathbf{0}$ always has at least one solution $\mathbf{x} = \mathbf{0}$.

The solution $\mathbf{x} = \mathbf{0}$ is called the **trivial solution**. A more interesting question for a homogeneous system is

Does it have a nontrivial solution?

Theorem

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the system has at least one free variable.

Example: Determine if the homogeneous system has a nontrivial solution. Describe the solution set.

(a)
$$\begin{array}{rcrcl} 2x_1 & + & x_2 & = & 0 \\ x_1 & - & 3x_2 & = & 0 \end{array}$$

We can use an augmented matrix w/ row reduction.

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & -3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

no free variables

The system has only the trivial solution

$x_1 = x_2 = 0$. In parametric form the solution set is described by

$x_1 = 0$
$x_2 = 0$

As a vector, this says $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Note: The coefficient matrix is

$$\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b)
$$\begin{array}{rcrcrcl} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}$$

Again using a matrix

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns (columns 1 and 2) — not a pivot column (column 3)

x_3 is a free variable $\Rightarrow$ there are ∞-many solutions.

$$x_1 \quad -4/3\, x_3 = 0$$
$$x_2 \quad = 0$$

The solution in parametric form is

$$x_1 = \frac{4}{3} x_3$$
$$x_2 = 0$$
$$x_3 \text{ is free}$$

As a vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

(c) $x_1 - 2x_2 + 5x_3 = 0$

$\begin{bmatrix} 1 & -2 & 5 & 0 \end{bmatrix}$ it's an rref

x_1 is basic, x_2, x_3 are free

$$x_1 = 2x_2 - 5x_3$$

x_2, x_3 - free

As a vector:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

Parametric Vector Form of a Solution Set

Example (b) had a solution set consisting of vectors of the form $\mathbf{x} = x_3\mathbf{v}$. Example (c)'s solution set consisted of vector that look like $\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v}$. Since these are **linear combinations**, we could write the solution sets like

$$\text{Span}\{\mathbf{v}\} \quad \text{or} \quad \text{Span}\{\mathbf{u}, \mathbf{v}\}.$$

Instead of using the variables x_2 and/or x_3 we often substitute **parameters** such as s or t.
The forms

$$\mathbf{x} = s\mathbf{u}, \quad \text{or} \quad \mathbf{x} = s\mathbf{u} + t\mathbf{v}$$

are called **parametric vector forms.**

Example

The **parametric vector form** of the solution set of
$x_1 \; - \; 2x_2 \; + \; 5x_3 \; = \; 0$ is

$$\mathbf{x} = s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, \quad \text{where } s, t \in \mathbb{R}.$$

Question: What geometric object is that solution set?