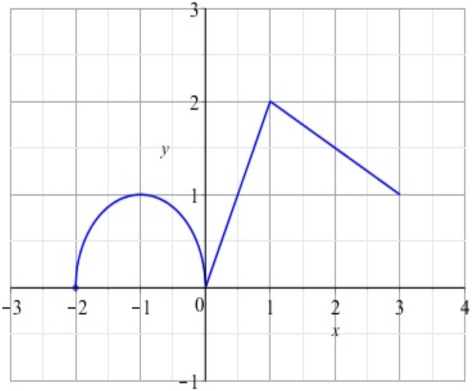
January 28 MATH 1112 sec. 54 Spring 2019

Section 2.5: Basic Transformations

From a small library of known function plots, we can graph a variety of functions if they can be determined as simple tranformations. We'll consider the following transformations:

- **Translations** shifting a graph up or down (vertical) or to the left or right (horizontal)
- **Reflections** taking the *mirror* image in the x or y axis
- **Scaling** stretching or shriking a graph in either of the vertical or horizontal orientations

Vertical Translation: $y = f(x) + b$ or $y = f(x) - b$



x	$f(x)$
-2	0
-1	1
0	0
1	2
2	$\frac{3}{2}$
3	1

Figure: The graph of $y = f(x)$ is shown along with a table of select points. Let's consider the plots of $y = f(x) + 1$ and $y = f(x) - 1$.

Vertical Translation: $y = f(x) + b$ or $y = f(x) - b$

x	$f(x)$
-2	0
-1	1
0	0
1	2
2	$\frac{3}{2}$
3	1

x	$f(x)+1$
-2	$f(-2)+1=0+1=1$
-1	$f(-1)+1 = 2$
0	1
1	3
2	$\frac{5}{2}$
3	2

x	$f(x)-1$
-2	$f(-2)-1 = 0-1=-1$
-1	$f(-1)-1 = 1-1 = 0$
0	-1
1	1
2	$\frac{1}{2}$
3	0

Figure: Complete the tables of values.

Vertical Translation: $y = f(x) + b$ or $y = f(x) - b$

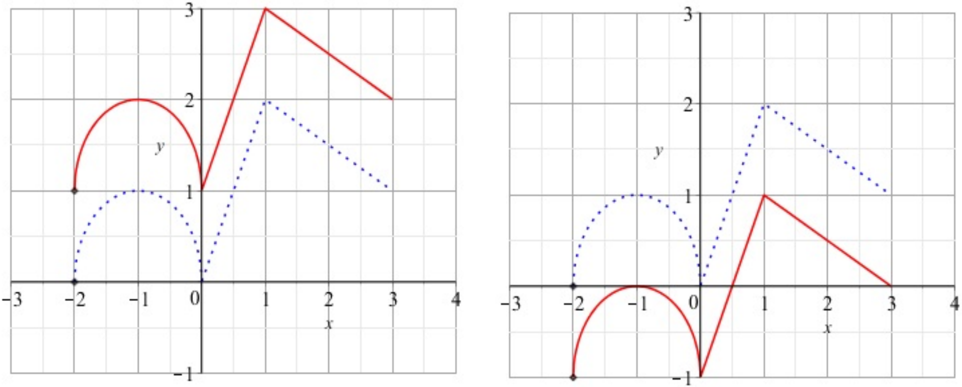
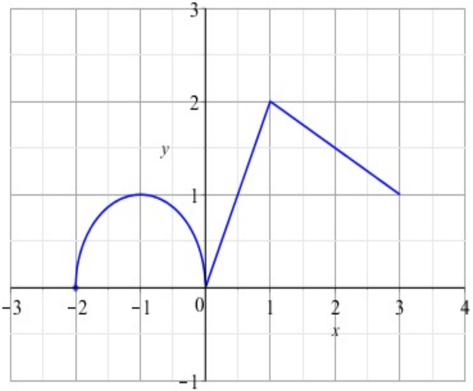


Figure: Left: $y = f(x)$ (blue dots), compared to $y = f(x) + 1$ (red)
Right: $y = f(x)$ (blue dots), compared to $y = f(x) - 1$ (red)

Horizontal Translation: $y = f(x - d)$ or $y = f(x + d)$



x	$f(x)$
-2	0
-1	1
0	0
1	2
2	$\frac{3}{2}$
3	1

Figure: The graph of $y = f(x)$ is shown along with a table of select points. Let's consider the plots of $y = f(x - 1)$ and $y = f(x + 1)$.

Horizontal Translation: $y = f(x - d)$ or $y = f(x + d)$

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	und	und	0	1	0	2	$\frac{3}{2}$	1	und
$f(x-1)$	f(-5) und	f(-4) und	f(-3) und	f(-2) 0	1	0	2	$\frac{3}{2}$	1
$f(x+1)$	f(-3) und	f(-2) 0	1	0	2	$\frac{3}{2}$	1	und	und

und = undefined

Figure: Complete the tables of values.

Horizontal Translation: $y = f(x - d)$ or $y = f(x + d)$

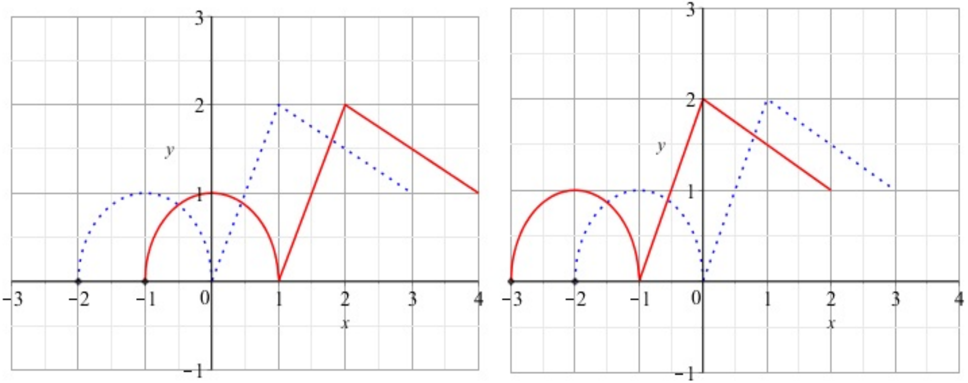


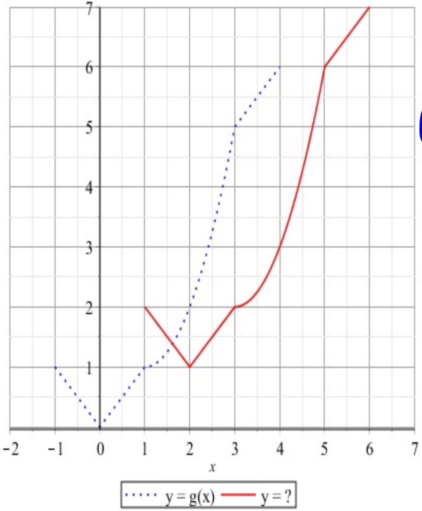
Figure: Left: $y = f(x)$ (blue dots), compared to $y = f(x - 1)$ (red)
Right: $y = f(x)$ (blue dots), compared to $y = f(x + 1)$ (red)

Vertical and Horizontal Translations

For $b > 0$ and $d > 0$

- the graph of $y = f(x) + b$ is the graph of $y = f(x)$ shifted up b units,
- the graph of $y = f(x) - b$ is the graph of $y = f(x)$ shifted down b units,
- the graph of $y = f(x - d)$ is the graph of $y = f(x)$ shifted right d units,
- the graph of $y = f(x + d)$ is the graph of $y = f(x)$ shifted left d units,

Question



The blue dotted curve is y = g(x). The red solid curve is the graph of y =

(a) $g(x-2)+1$

(b) $g(x+2)+1$

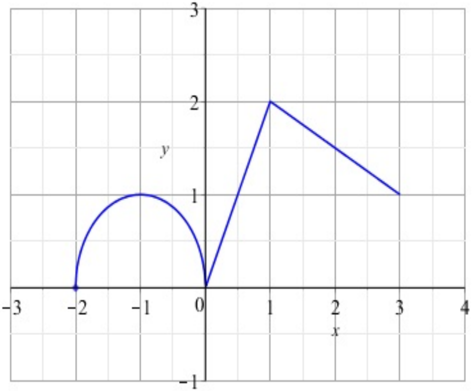
(c) $g(x-2)-1$

(d) $g(x+2)-1$

(e) can't be determined without more information

right 2-units and up 1-unit

Reflections: $y = f(-x)$ or $y = -f(x)$



x	$f(x)$
-2	0
-1	1
0	0
1	2
2	$\frac{3}{2}$
3	1

Figure: The graph of $y = f(x)$ is shown along with a table of select points. Now let's consider graphing $y = f(-x)$ and $y = -f(x)$

Reflections: $y = f(-x)$ or $y = -f(x)$

x	$f(x)$
-3	undef.
-2	0
-1	1
0	0
1	2
2	$\frac{3}{2}$
3	1

x	$f(-x)$
-3	$f(3) = 1$
-2	$f(2) = \frac{3}{2}$
-1	2
0	0
1	1
2	0
3	und

x	$-f(x)$
-3	$-f(-3)$ und
-2	$-f(-2) = -0 = 0$
-1	$-f(-1) = -1$
0	0
1	-2
2	$\frac{-3}{2}$
3	-1

Figure: Complete the tables of values.

Reflections: $y = f(-x)$ or $y = -f(x)$

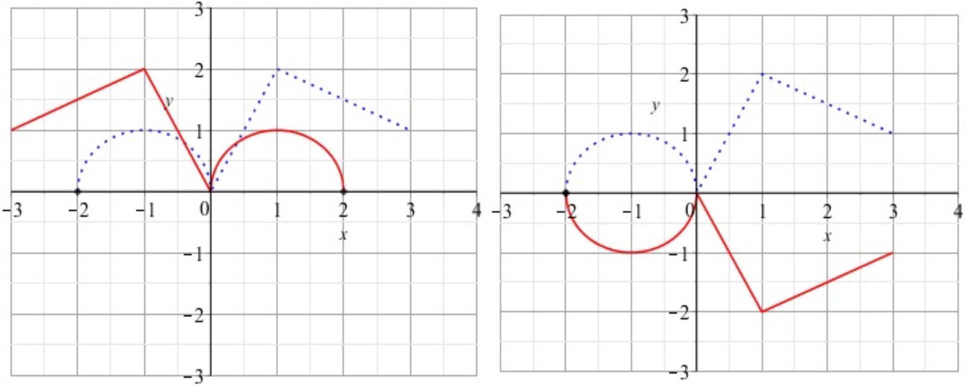


Figure: Left: $y = f(x)$ (blue dots), compared to $y = f(-x)$ (red)
Right: $y = f(x)$ (blue dots), compared to $y = -f(x)$ (red)

Reflection in the coordinate axes

The graph of $y = f(-x)$ is the reflection of the graph of $y = f(x)$ across the y-axis.

The graph of $y = -f(x)$ is the reflection of the graph of $y = f(x)$ across the x-axis.

Note that if (a, b) is a point on the graph of $y = f(x)$, then

(1) the point $(-a, b)$ is on the graph of $y = f(-x)$, and

(2) the point $(a, -b)$ is on the graph of $y = -f(x)$.

Stretching and Shrinking

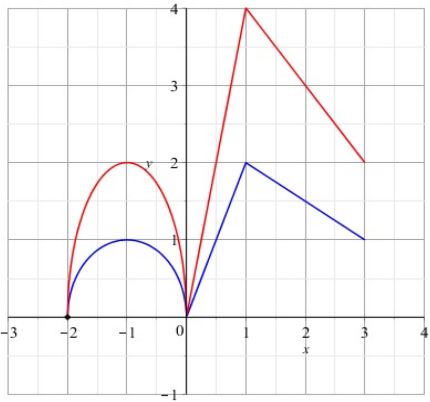
Since we already know that introducing a minus sign as in $f(-x)$ and $-f(x)$ results in a reflection, let's consider a positive number a and investigate the relationship between the graph of $y = f(x)$ and each of

$$y = af(x), \quad \text{and} \quad y = f(ax).$$

The outcome depends on whether $a > 1$ or $0 < a < 1$.

Why aren't we bothering with the case $a = 1$?

Vertical Stretch or Shrink: $y = af(x)$

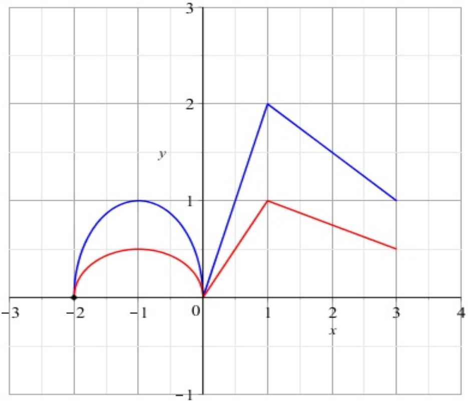


x	$f(x)$
-2	0
-1	1
0	0
1	2
2	$\frac{3}{2}$
3	1

x	$2f(x)$
-2	$2f(-2) = 2\cdot 0 = 0$
-1	$2f(-1) = 2\cdot 1 = 2$
0	0
1	4
2	3
3	2

Figure: $y = f(x)$ is in blue, and $y = 2f(x)$ is in red. Since $a = 2 > 1$, the graph is stretched vertically.

Vertical Stretch or Shrink: $y = af(x)$



x	$f(x)$
-2	0
-1	1
0	0
1	2
2	$\frac{3}{2}$
3	1

x	$\frac{1}{2}f(x)$
-2	$\frac{1}{2}f(-2) = 0$
-1	$\frac{1}{2}f(-1) = \frac{1}{2}$
0	0
1	1
2	$\frac{3}{4}$
3	$\frac{1}{2}$

Figure: $y = f(x)$ is in blue, and $y = \frac{1}{2}f(x)$ is in red. Since $a = \frac{1}{2} < 1$, the graph is shrinked vertically.

Vertical Stretch or Shrink: $y = af(x)$

The graph of $y = af(x)$ is obtained from the graph of $y = f(x)$. If $a > 0$, then

$y = af(x)$ is stretched vertically if $a > 1$, and
$y = af(x)$ is shrunk (a.k.a. compressed) vertically if $0 < a < 1$.

If $a < 0$, then the stretch ($|a| > 1$) or shrink ($0 < |a| < 1$) is combined with a reflection in the x-axis.

Horizontal Stretch or Shrink: $y = f(cx)$

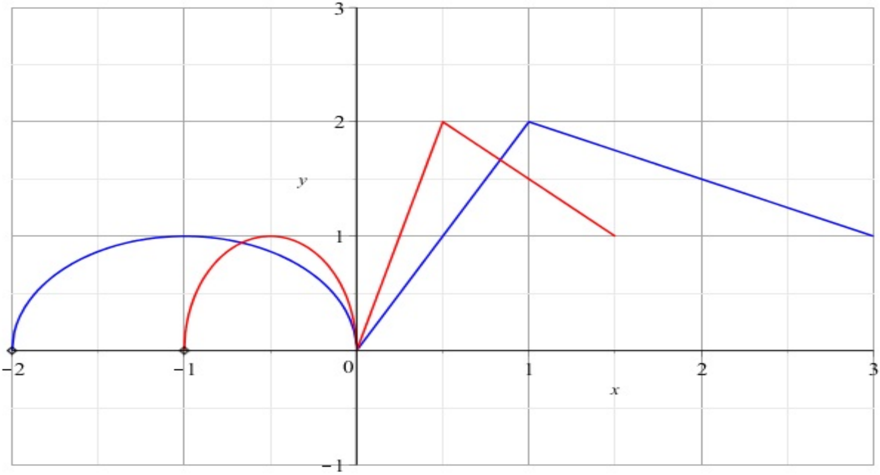


Figure: $y = f(x)$ is in blue, and $y = f(2x)$ is in red. Since $c = 2 > 1$, the graph is shrinked horizontally.

Horizontal Stretch or Shrink: $y = f(cx)$

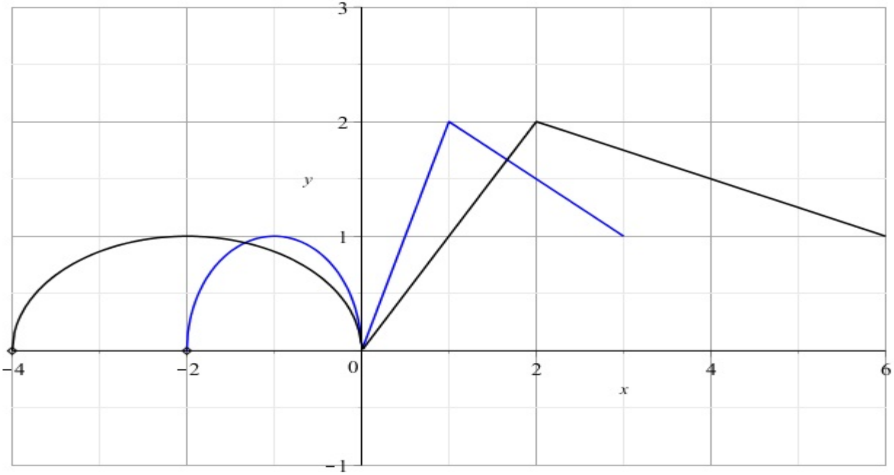


Figure: $y = f(x)$ is in blue, and $y = f\left(\frac{1}{2}x\right)$ is in black. Since $c = \frac{1}{2} < 1$, the graph is stretched horizontally.

Horizontal Stretch or Shrink: $y = f(cx)$

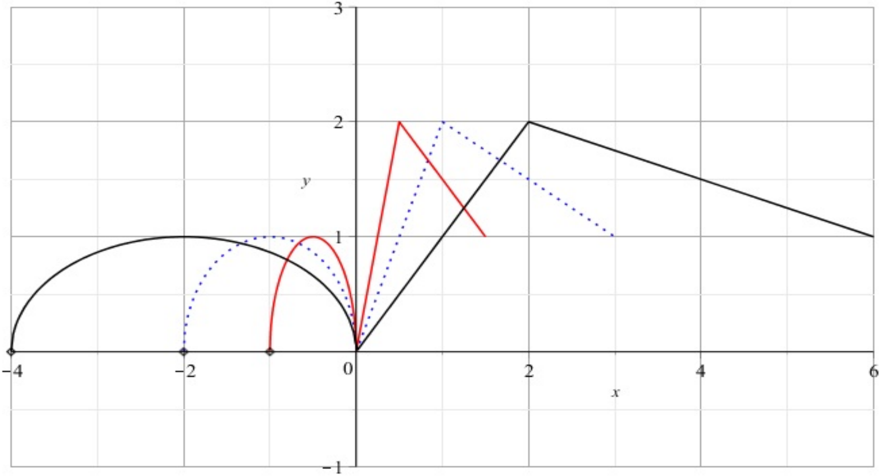


Figure: $y = f(x)$ is in blue dots. The compressed red curve is $y = f(2x)$, and the stretched black curve is $y = f\left(\frac{1}{2}x\right)$.

Horizontal Stretch or Shrink: $y = f(cx)$

The examples given generalize except that we did not consider an example with $c < 0$. This combines the stretch/shrink with a reflection. We have the following result:

The graph of $y = f(cx)$ is obtained from the graph of $y = f(x)$. If $c > 0$, then

$y = f(cx)$ is shrunk (a.k.a. compressed) horizontally if $c > 1$, and
$y = f(cx)$ is stretched horizontally if $0 < c < 1$.

If $c < 0$, then the shrink ($|c| > 1$) or stretch ($0 < |c| < 1$) is combined with a reflection in the y-axis.

Section 2.4: Symmetry

Consider the function $f(x) = 2x^2 + 1$. Suppose we wished to plot the new function $h(x) = f(-x)$. Note that

$$h(x) = f(-x) = 2(-x)^2 + 1 = 2x^2 + 1 = f(x).$$

Since the graph of h is obtained from f by reflection in the y-axis, and h and f are the same function, it must be that

the graph of f is its own reflection in the y-axis!

Definition: If a function f is called an **even function** if

$$f(-x) = f(x)$$

for each x in its domain. We can say that such a function has **even symmetry**.

Even Symmetry

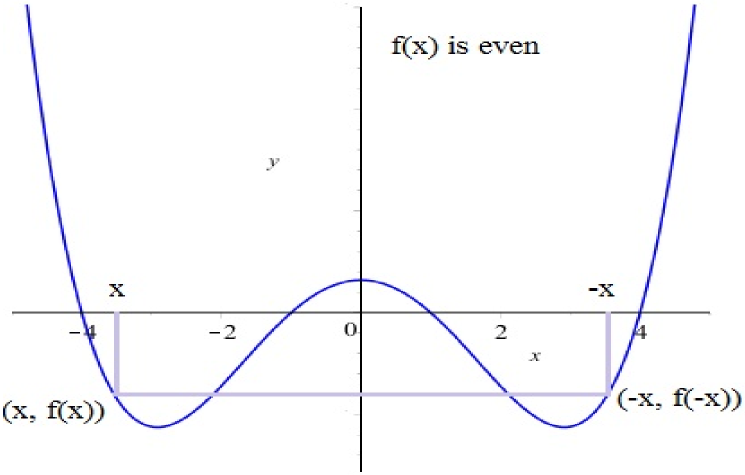


Figure: The graph to the left of the y-axis is the mirror image of the graph on the right side if a function is even.

Symmetry

Consider the function $f(x) = x - \frac{1}{2}x^3$, and let $g(x) = f(-x)$. Then note that

$$g(x) = f(-x) = (-x) - \frac{1}{2}(-x)^3 = -x + \frac{1}{2}x^3 = -\left(x - \frac{1}{2}x^3\right) = -f(x).$$

So $g(x)$ is the reflection in the y-axis, and it's equal to the reflection in the x-axis. That is

the reflection of f in the y-axis is its reflection in the x-axis!

Definition: If a function f is called an **odd function** if

$$f(-x) = -f(x)$$

for each x in its domain. We can say that such a function has **odd symmetry**.

Odd Symmetry

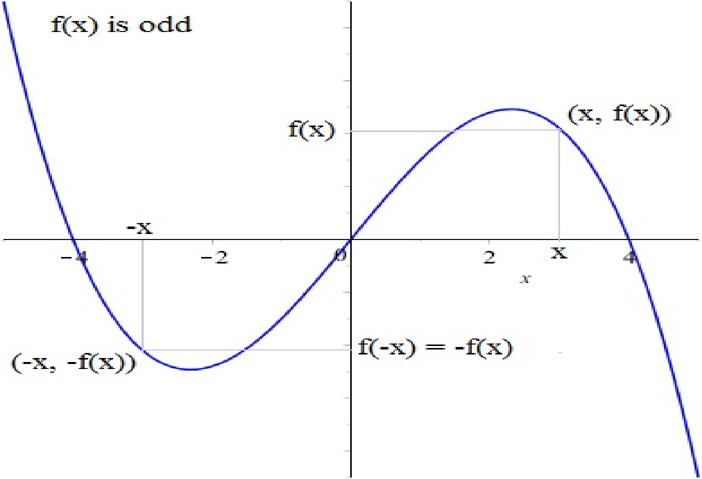


Figure: The graph of f to the left of the y-axis can be obtained by reflecting the graph on the right twice—through the y-axis and then the x-axis.