January 28 Math 2306 sec. 53 Spring 2019

Section 4: Exact Equations

If $M(x,y)\,dx + N(x,y)\,dy = 0$ happens to be exact, then it is equivalent to

$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 0$$

This implies that the function F is constant on R and solutions to the

DE are given by the relation

$$F(x,y) = C$$

Exact Equations

Theorem: Let M and N be continuous on some rectangle R in the plane. Then the equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Example

Show that the equation is exact and obtain a family of solutions.

$$(2xy - \sec^2 x)\,dx + (x^2 + 2y)\,dy = 0$$

$M(x,y) = 2xy - \sec^2 x$ and $N(x,y) = x^2 + 2y$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(2xy - \sec^2 x\right) = 2x - 0 = 2x$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + 2y\right) = 2x + 0 = 2x$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the ODE is exact

The solutions are given by $F(x,y) = C$

where $\dfrac{\partial F}{\partial x} = M(x,y) = 2xy - \sec^2 x$ and

$$\frac{\partial F}{\partial y} = N(x,y) = x^2 + 2y$$

When integrating note that

① the other variable is treated as a constant

② the "constant" of integration can depend on the other variable

$$F(x,y) = \int \frac{\partial F}{\partial x}\,dx = \int (2xy - \sec^2 x)\,dx$$

$$= 2y\left(\frac{x^2}{2}\right) - \tan x + g(y)$$

$$F(x,y) = x^2y - \tan x + g(y)$$

We need to find g. We know that

$$\frac{\partial F}{\partial y} = x^2 + 2y$$

Find $\frac{\partial F}{\partial y}$ above, and match it to this

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(x^2y - \tan x + g(y)\right)$$

$$= x^2 - 0 + g'(y)$$

Matching $\quad x^2 + 2y = x^2 + g'(y)$

So $\quad g'(y) = 2y$

One such $\quad g(y) = y^2$

$F(x,y) = x^2 y - \tan x + y^2$ $\quad$ up to an added constant

The solutions to the ODE are defined implicitly by $F(x,y) = C$ i.e.

$$x^2 y - \tan x + y^2 = C$$