January 28 Math 2306 sec. 60 Spring 2019

Section 4: Exact Equations

If $M(x,y)\,dx + N(x,y)\,dy = 0$ happens to be exact, then it is equivalent to

$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 0$$

This implies that the function F is constant on R and solutions to the

DE are given by the relation

$$F(x,y) = C$$

Exact Equations

Theorem: Let M and N be continuous on some rectangle R in the plane. Then the equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Example

Show that the equation is exact and obtain a family of solutions.

$$(2xy - \sec^2 x)\,dx + (x^2 + 2y)\,dy = 0$$

$M(x,y) = 2xy - \sec^2 x$ and $N(x,y) = x^2 + 2y$

$\frac{\partial M}{\partial y} = 2x - 0 = 2x$ $\frac{\partial N}{\partial x} = 2x + 0 = 2x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ the ODE is exact.

We seek $F(x,y)$ such that

$\frac{\partial F}{\partial x} = 2xy - \sec^2 x$ and

$\frac{\partial F}{\partial y} = x^2 + 2y$

To find F we can integrate $\frac{\partial F}{\partial x}$ with respect to x.

$$F(x,y) = \int \frac{\partial F}{\partial x}\,dx = \int \left(2xy - \sec^2 x\right) dx$$

2 points:

(i) We hold y constant while integrating with respect to x

(ii) The "constant" of integration can depend on the other variable (here y)

$$F(x,y) = \int (2xy - \sec^2 x)\,dx$$

$$= 2y\left(\frac{x^2}{2}\right) - \tan x + g(y)$$

"constant of integration"

$$F(x,y) = x^2 y - \tan x + g(y).$$

We need to find g. We know that

$$\frac{\partial F}{\partial y} = N(x,y) = x^2 + 2y$$

We'll take $\frac{\partial}{\partial y}$ of F and match.

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(x^2 y - \tan x + g(y)\right)$$

$$= x^2 - 0 + g'(y)$$

Match

$$x^2 + g'(y) = x^2 + 2y$$

$$g'(y) = 2y$$

One function $g(y) = y^2$

So
$$F(x,y) = x^2 y - \tan x + y^2$$

(up to added constant)

The solutions to the ODE are given implicitly by the relation $F(x,y) = C$. i.e.

$$x^2 y - \tan x + y^2 = C$$