# January 28 Math 2335 sec 51 Spring 2016

## Section 2.3: Propagation of Error

Suppose $x_A$ and $y_A$ are approximations to the true quantities $x_T$ and $y_T$, respectively.

**Question:** If we use the approximate values to compute approximations to $x_T + y_T$, $x_T - y_T$, $x_T \times y_T$ or $x_T \div y_T$, how will the errors in $x_A$ and $y_A$ affect the error in the result. For example, how does

$$\mathrm{Err}(x_A \times y_A) = (x_T \times y_T) - (x_A \times y_A)$$

depend on $\mathrm{Err}(x_A)$ and $\mathrm{Err}(y_A)$?

We'll call the error $\mathrm{Err}(x_A \times y_A)$ the *propagated error* and usually denote it $E$.

# Interval Arithmetic

Some useful results are the triangle inequalities:

$$|A+B| \leq |A| + |B|, \quad \text{and} \quad |A-B| \leq |A| + |B|$$

$$||A| - |B|| \leq |A - B|$$

# Bounding Propagated Error: Interval Arithmetic

For $x_A = 3.14$ and $y_A = 2.718$ correctly rounded to the digits shown, bound the absolute value of the propagated error $|E|$ for the division

$$E = \frac{x_T}{y_T} - \frac{x_A}{y_A}.$$

Recall from last time that we deteremined that

$$3.135 \le x_T < 3.145, \quad \text{and} \quad 2.7175 \le y_T < 2.7185$$

so that

$$|\mathrm{Err}(x_A)| \le 0.005 \quad \text{and} \quad |\mathrm{Err}(y_A)| \le 0.0005.$$

For $\quad 2.7175 \le y_T < 2.7185$

$$\frac{1}{2.7185} < \frac{1}{y_T} \le \frac{1}{2.7175}$$

So

$$\frac{3.135}{2.7185} \leq \frac{x_T}{y_T} \leq \frac{3.145}{2.7175}$$

$$-\frac{x_A}{y_A} \qquad -\frac{x_A}{y_A} \qquad -\frac{x_A}{y_A}$$

$$\frac{3.135}{2.7185} - \frac{3.14}{2.718} \leq \frac{x_T}{y_T} - \frac{x_A}{y_A} \leq \frac{3.145}{2.7175} - \frac{3.14}{2.718}$$

$$-0.002052 \leq E \leq 0.002053$$

$$-0.002053 < E \leq 0.002053$$

$$|E| \leq 0.002053$$

# Propagated Error in Multiplication

Let $x_A$ and $y_A$ approximate $x_T$ and $y_T$, respectively. Show that if $\text{Rel}(x_A)$ and $\text{Rel}(y_A)$ are very small (compared to 1), then

$$\text{Rel}(x_A y_A) \approx \text{Rel}(x_A) + \text{Rel}(y_A).$$

$$\text{Rel}(x_A y_A) = \frac{x_T y_T - x_A y_A}{x_T y_T}$$

$$\text{Err}(x_A) = x_T - x_A \implies x_A = x_T - \text{Err}(x_A)$$

similarly $\quad y_A = y_T - \text{Err}(y_A)$

$$Rel(x_A y_A) = \frac{x_T y_T - (x_T - Err(x_A))(y_T - Err(y_A))}{x_T y_T}$$

$$= \frac{1}{x_T y_T}\left[x_T y_T - \left(x_T y_T - y_T Err(x_A) - x_T Err(y_A) + Err(x_A) Err(y_A)\right)\right]$$

$$= \frac{1}{x_T y_T}\left[x_T y_T - x_T y_T + y_T Err(x_A) + x_T Err(y_A) - Err(x_A) Err(y_A)\right]$$

$$= \frac{1}{x_T y_T}\left[y_T Err(x_A) + x_T Err(y_A) - Err(x_A) Err(y_A)\right]$$

$$= \frac{y_T \, Err(x_A)}{x_T y_T} + \frac{x_T \, Err(y_A)}{x_T y_T} - \frac{Err(x_A)\, Err(y_A)}{x_T y_T}$$

So

$$Rel(x_A y_A) = \frac{Err(x_A)}{x_T} + \frac{Err(y_A)}{y_T} - \left(\frac{Err(x_A)}{x_T}\right)\left(\frac{Err(y_A)}{y_T}\right)$$

$$Rel(x_A y_A) = Rel(x_A) + Rel(y_A) - Rel(x_A)\, Rel(y_A)$$

For $|Rel(x_A)| << 1$ and $|Rel(y_A)| << 1$

The product $Rel(x_A)\,Rel(y_A)$ is negligible.

Thus

$$Rel(x_A y_A) \approx Rel(x_A) + Rel(y_A)$$

# Mean Value Theorem (for the derivative)

**Theorem:** Suppose $f$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval $(a, b)$. Then there exists a number $c$ in $(a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This gives $\quad f(b) - f(a) = f'(c)\,(b-a)$

# Propagated Error in Function Evaluation

Let $f(x)$ be some differentiable function. Suppose $x_A$ is an approximation to $x_T$, and we wish to determine the function value $f(x_T)$. We may approximate $f(x_T)$ with $f(x_A)$. This contains propagated error that depends both on $\mathrm{Err}(x_A)$ and on the nature of the function $f$.

From the Mean Value Theorem, we have

$$E = f(x_T) - f(x_A) = f'(c)(x_T - x_A)$$

where $c$ is some number between $x_T$ and $x_A$.

As usual, the value $c$ is not known. If $x_T \approx x_A$ (as should be the case), we may approximate $f'(c)$ with $f'(x_T)$ or with $f'(x_A)$—we do at least know what $x_A$ is.

# Example

Assume $x_A = 2.62$ is correctly rounded to the digits shown. Bound the error $|f(x_T) - f(x_A)|$ and the relative error $\mathrm{Rel}(f(x_A))$ in the approximation $\tan^{-1}(2.62)$.

Here, $f(x) = \tan^{-1} x$

Since $x_A$ is correctly rounded

$$2.615 \le x_T < 2.625$$

From the MVT

$$f(x_T) - f(x_A) = f'(c)(x_T - x_A)$$

For some $c$ between $x_T$ and $x_A$

$$2.615 - 2.62 \le x_T - x_A < 2.625 - 2.62$$

$$\Rightarrow \quad |x_T - x_A| \le 0.005$$

$$f'(x) = \frac{1}{1+x^2} \quad \text{so} \quad f'(c) = \frac{1}{1+c^2}$$

For $\quad 2.615 \le c < 2.625$

$$(2.615)^2 \le c^2 < (2.625)^2$$

$$1 + (2.615)^2 \le 1 + c^2 < 1 + (2.625)^2$$

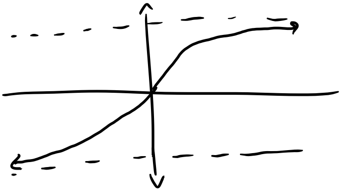
$$\frac{1}{1+(2.625)^2} \le \frac{1}{1+c^2} \le \frac{1}{1+(2.615)^2} \doteq 0.12758$$

$$|\tan^{-1} x_T - \tan^{-1} x_A| \le 0.12758 \cdot 0.005 \doteq 0.000638$$

$$\mathrm{Rel}(\tan^{-1}(x_A)) = \frac{E}{\tan^{-1}(x_T)}$$

For $2.615 \le x_T < 2.625$

Recall $\tan^{-1}x$ is an increasing function.



$$\tan^{-1}(2.615) \le \tan^{-1}(x_T) < \tan^{-1}(2.625)$$

$$\Rightarrow \quad \frac{1}{\tan^{-1}(2.625)} \le \frac{1}{\tan^{-1}(x_T)} \le \frac{1}{\tan^{-1}(2.615)}$$

$$\left| Rel(\tan^{-1} x_T) \right| = \left| \frac{E}{\tan^{-1}(x_T)} \right| \le 0.000638 \quad \frac{1}{\tan^{-1}(2.615)}$$

$$\doteq 0.000529$$

# Propagated Error Estimates in Function Evaluation

We have the propagated error and relative error estimates

$$E \quad \approx \quad f'(x_T)\mathrm{Err}(x_A),$$

and

$$\mathrm{Rel}(f(x_A)) \approx \frac{f'(x_T)(x_T - x_A)}{f(x_T)} = \frac{x_T f'(x_T)}{f(x_T)}\mathrm{Rel}(x_A).$$

# Example

Let $f(x) = b^x$ for some positive number $b \neq 1$. Find expressions for the propagated error and relative error $f(x_T) - f(x_A)$ and $\mathrm{Rel}(f(x_A))$.

$$f(x_T) - f(x_A) \approx f'(x_T)(x_T - x_A)$$

$$f'(x) = b^x \ln b$$ Hence

$$E \approx b^{x_T} \ln b \, (x_T - x_A)$$

$$Rel(f(x_A)) \approx \frac{x_T f'(x_T)}{f(x_T)} Rel(x_A)$$

$$= \frac{x_T b^{x_T} \ln b}{b^{x_T}} Rel(x_A)$$

$$= x_T \ln b \; Rel(x_A)$$

# Example

Approximate the error and relative error when $2^{22/7}$ is used to approximate $2^{\pi}$.

Our $b=2$ $\quad x_T = \pi \quad x_A = \frac{22}{7}$

$$E \approx 2^{\pi} \ln 2 \left(\pi - \frac{22}{7}\right) \approx 2^{22/7} \ln 2 \left(\pi - \frac{22}{7}\right)$$

$$\doteq -0.007735$$

$$\text{Rel}\left(2^{22/7}\right) = \pi \ln 2 \ \text{Rel}(x_A) \doteq -0.000876$$