# January 29 Math 3260 sec. 55 Spring 2020

## Section 1.5: Solution Sets of Linear Systems

**Definition** A linear system is said to be **homogeneous** if it can be written in the form

$$A\mathbf{x} = \mathbf{0}$$

for some $m \times n$ matrix $A$ and where $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.

**Theorem:** A homogeneous system $A\mathbf{x} = \mathbf{0}$ always has at least one solution $\mathbf{x} = \mathbf{0}$.

The solution $\mathbf{x} = \mathbf{0}$ is called the **trivial solution**.

## Theorem

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the system has at least one free variable.

We considered the example (last time) **Example:** Determine if the homogeneous system has a nontrivial solution. Describe the solution set.

(b)
$$\begin{array}{rcrcrcl} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}$$

Using an augmented matrix, we got

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \quad \text{rref} \longrightarrow \quad \begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# Example Continued...

From the rref, we see that $x_1$ and $x_2$ are basic, and $x_3$ is free giving us infinitely many solutions that can be expressed in

**Parametric Form:**
$$\begin{array}{lcl} x_1 & = & \frac{4}{3}x_3 \\ x_2 & = & 0 \\ x_3 & \text{is} & \text{free} \end{array}$$

or in

**Parametric Vector Form:** $$\mathbf{x} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix},$$

where the free variable, $x_3$ can be any real number.

# Nonhomogeneous Systems

Find all solutions of the nonhomogeneous system of equations

$$\begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & 4x_3 & = & 7 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & -1 \\ 6x_1 & + & x_2 & - & 8x_3 & = & -4 \end{array}$$

Using an augmented matrix

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The solutions satisfy

$$x_1 = -1 + \frac{4}{3}x_3$$

$$x_2 = 2$$

$x_3$ - free

In vector form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

This has the form of a constant vector plus any solution to the system that is homogeneous with the same left hand side.

# Solutions of Nonhomogeneous Systems

Note that the solution in this example has the form

$$\mathbf{x} = \mathbf{p} + t\mathbf{v}$$

with **p** and **v** fixed vectors and $t$ a varying parameter. Also note that the $t\mathbf{v}$ part is the solution to the previous example with the right hand side all zeros. This is no coincidence!

**p** is called a **particular solution**, and $t\mathbf{v}$ is called a solution to the associated homogeneous equation.

# Theorem

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for a given $\mathbf{b}$. Let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form

$$\mathbf{x} = \mathbf{p} + \mathbf{v}_h,$$

where $\mathbf{v}_h$ is any solution of the associated homogeneous equation $A\mathbf{x} = \mathbf{0}$.

We can use a row reduction technique to get all parts of the solution in one process.

# Example

Find the solution set of the following system. Express the solution set in parametric vector form.

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & - & 2x_3 & + & 4x_4 & = & 1 \\ 2x_1 & + & 3x_2 & - & 6x_3 & + & 12x_4 & = & 4 \end{array}$$

Using an augmented matrix

$$\begin{bmatrix} 1 & 1 & -2 & 4 & 1 \\ 2 & 3 & -6 & 12 & 4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -2 & 4 & 2 \end{bmatrix}$$

$x_1 = -1$

$x_2 = 2 + 2x_3 - 4x_4$

$x_3, x_4$ are free

In parametric vector form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2+2x_3-4x_4 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}}_{\vec{p}} + \underbrace{x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -4 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_h}$$

# Section 1.7: Linear Independence

We already know that a homogeneous equation $A\mathbf{x} = \mathbf{0}$ can be thought of as an equation in the column vectors of the matrix $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots x_n\mathbf{a}_n = \mathbf{0}.$$

And, we know that at least one solution (the trivial one $x_1 = x_2 = \cdots = x_n = 0$) always exists.

Whether or not there is a nontrivial solution gives us a way to characterize the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

# Definition: Linear Dependence/Independence

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

The set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists a set of weights $c_1, c_2, \dots, c_p$ *at least one of which is nonzero* such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}.$$

(i.e. Provided the homogeneous equation posses a nontrivial solution.)

An equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}$, with at least one $c_i \neq 0$, is called a **linear dependence relation**.

# Theorem on Linear Independence

**Theorem:** The columns of a matrix $A$ are linearly **independent** if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

# Example

Determine if the set is linearly dependent or linearly independent.

(a) $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

We can use a homogeneous system $A\vec{x} = \vec{0}$ by letting $A = [\vec{v}_1 \ \vec{v}_2]$.

* $A\vec{x} = \vec{0}$ is equivalent to $x_1\vec{v}_1 + x_2\vec{v}_2 = \vec{0}$

Using an augmented matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 4 & -2 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

no free variables

$A\vec{x} = \vec{0}$ has only the trivial solution. Hence $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent.

# Example

Determine if the set is linearly dependent or linearly independent.

(b) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Again consider $A\vec{x} = \vec{0}$ where $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

↑ non pivot column ⇒ Free variable

Due to a free variable, $A\vec{x} = \vec{0}$ has nontrivial solutions. Hence $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.

# Example

Determine if the set of vectors is linearly dependent or independent. If dependent, find a linear dependence relation.

(c) $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}$

label $\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \quad \vec{v}_4$

we'll use an augmented matrix

$A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4]$

$A\vec{x} = \vec{0}$ is equivalent to

$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 = \vec{0}$

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 0 & 2 & 3 & 2 & 0 \\ 0 & 1 & 3 & 0 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

↑ free variable

$A\vec{x}=\vec{0}$ has nontrivial solutions, the vectors are linearly dependent.

Note, from the rref

$x_1 = -\frac{1}{3}x_4$

$x_2 = -2x_4$ $\quad$ $x_4$ is free

$x_3 = \frac{2}{3}x_4$

The vector equation is

$$-\frac{1}{3}x_4\vec{v}_1 - 2x_4\vec{v}_2 + \frac{2}{3}x_4\vec{v}_3 + x_4\vec{v}_4 = \vec{0}$$

Taking $x_4 = 1$, we get a linear dependence relation

$$-\frac{1}{3}\vec{v}_1 - 2\vec{v}_2 + \frac{2}{3}\vec{v}_3 + \vec{v}_4 = \vec{0}$$

# Theorem

An indexed set of two or more vectors is linearly dependent if and only if at least one vector in the set is a linear combination of the others in the set.

**Example:** Let **u** and **v** be any nonzero vectors in $\mathbb{R}^3$. Show that if **w** is any vector in Span$\{\mathbf{u}, \mathbf{v}\}$, then the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly **dependent.**

Since $\vec{w}$ is in Span$\{\vec{u}, \vec{v}\}$, there are some numbers $c_1, c_2$ such that

$$\vec{w} = c_1\vec{u} + c_2\vec{v}$$

Subtracting $\vec{w}$, we get

$$c_1\vec{u} + c_2\vec{v} - \vec{w} = \vec{0}$$

This is a linear dependence relation.

The coefficient of $\vec{w}$ is $-1$ and

$-1 \neq 0$.

# Caveat!

A set may be linearly dependent even if all proper subsets are linearly independent. For example, consider

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Each set $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, and $\{\mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. (You can easily verify this.)

However,

$$\mathbf{v}_3 = \mathbf{v}_2 - \mathbf{v}_1 \quad \text{i.e.} \quad \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0},$$

so the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

# Two More Theorems

**Theorem:** If a set contains more vectors than there are entries in each vector, then the set is linearly **dependent.** That is, if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a set of vector in $\mathbb{R}^n$, and $p > n$, then the set is linearly dependent.

For example, 4 vectors in $\mathbb{R}^3$ are necessarily lin. dependent.

**Theorem:** Any set of vectors that contains the zero vector is linearly **dependent.**